MATH 409

Advanced Calculus I

Lecture 19:
Riemann sums.
Properties of integrals.

Darboux sums

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of an interval $[a, b]$, where $x_0 = a < x_1 < \cdots < x_n = b$. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

Definition. The **upper Darboux sum** (or the **upper Riemann sum**) of the function f over the partition P is the number

$$U(f, P) = \sum_{j=1}^{n} M_j(f)\, \Delta_j,$$

where $\Delta_j = x_j - x_{j-1}$ and $M_j(f) = \sup f([x_{j-1}, x_j])$ for $j = 1, 2, \ldots, n$. Likewise, the **lower Darboux sum** (or the **lower Riemann sum**) of f over P is the number

$$L(f, P) = \sum_{j=1}^{n} m_j(f)\, \Delta_j,$$

where $m_j(f) = \inf f([x_{j-1}, x_j])$ for $j = 1, 2, \ldots, n$.

Upper and lower integrals

Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function.

Definition. The **upper integral** of f on $[a, b]$, denoted $\overline{\int_a^b} f(x)\,dx$ or $(U)\int_a^b f(x)\,dx$, is the number

$$\inf \{ U(f, P) \mid P \text{ is a partition of } [a, b] \}.$$

Similarly, the **lower integral** of f on $[a, b]$, denoted $\underline{\int_a^b} f(x)\,dx$ or $(L)\int_a^b f(x)\,dx$, is the number

$$\sup \{ L(f, P) \mid P \text{ is a partition of } [a, b] \}.$$

Remark. For any partitions P and Q of the interval $[a, b]$,

$$L(f, P) \leq (L)\int_a^b f(x)\,dx \leq (U)\int_a^b f(x)\,dx \leq U(f, Q).$$

Integrability

Definition. A bounded function $f : [a, b] \to \mathbb{R}$ is called **integrable** (or **Riemann integrable**) on the interval $[a, b]$ if the upper and lower integrals of f on $[a, b]$ coincide. The common value is called the **integral** of f on $[a, b]$ (or over $[a, b]$).

Theorem A bounded function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there is a partition P_ε of $[a, b]$ such that $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$.

Theorem If a function is continuous on the interval $[a, b]$, then it is integrable on $[a, b]$.

Riemann sums

Definition. A **Riemann sum** of a function $f : [a, b] \to \mathbb{R}$ with respect to a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ generated by samples $t_j \in [x_{j-1}, x_j]$ is a sum

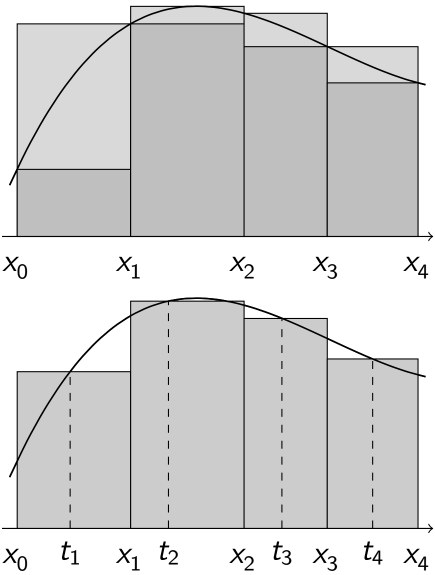
$$\mathcal{S}(f, P, t_j) = \sum_{j=1}^{n} f(t_j)\,(x_j - x_{j-1}).$$

Remark. Note that the function f need not be bounded. If f is bounded, then $L(f, P) \le \mathcal{S}(f, P, t_j) \le U(f, P)$ for any choice of samples t_j.

Definition. The Riemann sums $\mathcal{S}(f, P, t_j)$ **converge** to a limit $I(f)$ as the norm $\|P\| \to 0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P\| < \delta$ implies $|\mathcal{S}(f, P, t_j) - I(f)| < \varepsilon$ for any partition P and choice of samples t_j.

Theorem The Riemann sums $\mathcal{S}(f, P, t_j)$ converge to a limit $I(f)$ as $\|P\| \to 0$ if and only if the function f is integrable on $[a, b]$ and $I(f) = \int_a^b f(x)\,dx$.

Darboux sums and a Riemann sum



Proof of the theorem ("only if"): Assume that the Riemann sums $\mathcal{S}(f, P, t_j)$ converge to a limit $I(f)$ as $\|P\| \to 0$. Given $\varepsilon > 0$, we choose $\delta > 0$ so that for every partition P with $\|P\| < \delta$, we have $|\mathcal{S}(f, P, t_j) - I(f)| < \varepsilon$ for any choice of samples t_j. Let $\tilde{t}_j$ be a different set of samples for the same partition P. Then $|\mathcal{S}(f, P, \tilde{t}_j) - I(f)| < \varepsilon$. We can choose the samples $t_j, \tilde{t}_j$ so that $f(t_j)$ is arbitrarily close to $\sup f([x_{j-1}, x_j])$ while $f(\tilde{t}_j)$ is arbitrarily close to $\inf f([x_{j-1}, x_j])$. That way $\mathcal{S}(f, P, t_j)$ gets arbitrarily close to $U(f, P)$ while $\mathcal{S}(f, P, \tilde{t}_j)$ gets arbitrarily close to $L(f, P)$. Hence it follows from the above inequalities that $|U(f, P) - I(f)| \le \varepsilon$ and $|L(f, P) - I(f)| \le \varepsilon$. As a consequence, $U(f, P) - L(f, P) \le 2\varepsilon$. In particular, the function f is bounded. We conclude that f is integrable.

Let $I = \int_a^b f(x)\,dx$. The number I lies between $L(f, P)$ and $U(f, P)$. The inequalities $U(f, P) - L(f, P) \le 2\varepsilon$ and $|U(f, P) - I(f)| \le \varepsilon$ imply that $|I - I(f)| \le 3\varepsilon$. As ε can be arbitrarily small, $I = I(f)$.

Integration as a linear operation

Theorem 1 If functions f, g are integrable on an interval $[a, b]$, then the sum $f + g$ is also integrable on $[a, b]$ and

$$\int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Theorem 2 If a function f is integrable on $[a, b]$, then for each $\alpha \in \mathbb{R}$ the scalar multiple αf is also integrable on $[a, b]$ and

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx.$$

Proof of Theorems 1 and 2: Let $I(f)$ denote the integral of f and $I(g)$ denote the integral of g over $[a, b]$. The key observation is that the Riemann sums depend linearly on a function. Namely, $\mathcal{S}(f+g, P, t_j) = \mathcal{S}(f, P, t_j) + \mathcal{S}(g, P, t_j)$ and $\mathcal{S}(\alpha f, P, t_j) = \alpha \cdot \mathcal{S}(f, P, t_j)$ for any partition P of $[a, b]$ and choice of samples t_j. It follows that

$$|\mathcal{S}(f+g, P, t_j) - I(f) - I(g)| \\ \leq |\mathcal{S}(f, P, t_j) - I(f)| + |\mathcal{S}(g, P, t_j) - I(g)|,$$

$$|\mathcal{S}(\alpha f, P, t_j) - \alpha I(f)| = |\alpha| \cdot |\mathcal{S}(f, P, t_j) - I(f)|.$$

As $\|P\| \to 0$, the Riemann sums $\mathcal{S}(f, P, t_j)$ and $\mathcal{S}(g, P, t_j)$ get arbirarily close to $I(f)$ and $I(g)$, respectively. Then $\mathcal{S}(f+g, P, t_j)$ will be getting arbitrarily close to $I(f) + I(g)$ while $\mathcal{S}(\alpha f, P, t_j)$ will be getting arbitrarily close to $\alpha I(f)$. Thus $I(f) + I(g)$ is the integral of $f + g$ and $\alpha I(f)$ is the integral of αf over $[a, b]$.

Theorem If a function f is integrable on $[a, b]$, then it is integrable on each subinterval $[c, d] \subset [a, b]$.

Proof: Since f is integrable on the interval $[a, b]$, for any $\varepsilon > 0$ there is a partition P_ε of $[a, b]$ such that $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$. Given a subinterval $[c, d] \subset [a, b]$, let $P'_\varepsilon = P_\varepsilon \cup \{c, d\}$ and $Q_\varepsilon = P'_\varepsilon \cap [c, d]$. Then P'_ε is a partition of $[a, b]$ that refines P_ε. Hence

$$U(f, P'_\varepsilon) - L(f, P'_\varepsilon) \leq U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

Since Q_ε is a partition of $[c, d]$ contained in P'_ε, it follows that

$$U(f, Q_\varepsilon) - L(f, Q_\varepsilon) \leq U(f, P'_\varepsilon) - L(f, P'_\varepsilon) < \varepsilon.$$

We conclude that f is integrable on $[c, d]$.

Theorem If a function f is integrable on $[a, b]$ then for any $c \in (a, b)$,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof: Since f is integrable on the interval $[a, b]$, it is also integrable on subintervals $[a, c]$ and $[c, b]$. Let P be a partition of $[a, c]$ and $\{t_j\}$ be some samples for that partition. Further, let Q be a partition of $[c, b]$ and $\{\tau_i\}$ be some samples for that partition. Then $P \cup Q$ is a partition of $[a, b]$ and $\{t_j\} \cup \{\tau_i\}$ are samples for it. The key observation is that

$$\mathcal{S}(f, P \cup Q, \{t_j\} \cup \{\tau_i\}) = \mathcal{S}(f, P, t_j) + \mathcal{S}(f, Q, \tau_i).$$

If $\|P\| \to 0$ and $\|Q\| \to 0$, then $\|P \cup Q\| = \max(\|P\|, \|Q\|)$ tends to 0 as well. Therefore the Riemann sums in the latter equality will converge to the integrals $\int_a^b f(x)\,dx$, $\int_a^c f(x)\,dx$, and $\int_c^b f(x)\,dx$, respectively.

Theorem If a function f is integrable on $[a, b]$ and $f([a, b]) \subset [A, B]$, then for each continuous function $g : [A, B] \to \mathbb{R}$ the composition $g \circ f$ is also integrable on $[a, b]$.

Corollary If functions f and g are integrable on $[a, b]$, then so is fg.

Proof: We have $(f + g)^2 = f^2 + g^2 + 2fg$. Since f and g are integrable on $[a, b]$, so is $f + g$. Since $h(x) = x^2$ is a continuous function on $\mathbb{R}$, the compositions $h \circ f = f^2$, $h \circ g = g^2$, and $h \circ (f + g) = (f + g)^2$ are integrable on $[a, b]$. Then $fg = \frac{1}{2}(f + g)^2 - \frac{1}{2}f^2 - \frac{1}{2}g^2$ is integrable on $[a, b]$ as a linear combination of integrable functions.

Comparison Theorem for integrals

Theorem If functions f, g are integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$

Proof: Since $f \leq g$ on the interval $[a, b]$, it follows that $\mathcal{S}(f, P, t_j) \leq \mathcal{S}(g, P, t_j)$ for any partition P of $[a, b]$ and choice of samples t_j. As $\|P\| \to 0$, the sum $\mathcal{S}(f, P, t_j)$ gets arbitrarily close to the integral of f while $\mathcal{S}(g, P, t_j)$ gets arbitrarily close to the integral of g. The theorem follows.

Corollary 1 If f is integrable on $[a, b]$ and $f(x) \geq 0$ for $x \in [a, b]$, then $\displaystyle\int_a^b f(x)\, dx \geq 0$.

Corollary 2 If f is integrable on $[a, b]$ and $m \leq f(x) \leq M$ for $x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Corollary 3 If f is integrable on $[a, b]$, then the function $|f|$ is also integrable on $[a, b]$ and

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Proof: The function $|f|$ is the composition of f with a continuous function $g(x) = |x|$. Therefore $|f|$ is integrable on $[a, b]$. Since $-|f(x)| \leq f(x) \leq |f(x)|$ for $x \in [a, b]$, the Comparison Theorem for integrals implies that

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx.$$

Integral with variable limit

Suppose $f : [a, b] \to \mathbb{R}$ is an integrable function. For any $x \in [a, b]$ let $F(x) = \int_a^x f(t)\,dt$ (we assume that $F(a) = 0$).

Theorem The function F is well defined and continuous on $[a, b]$.

Proof: Since the function f is integrable on $[a, b]$, it is also integrable on each subinterval of $[a, b]$. Hence the function F is well defined on $[a, b]$. Besides, f is bounded: $|f(t)| \leq M$ for some $M > 0$ and all $t \in [a, b]$. For any $x, y \in [a, b]$, $x \leq y$, we have $\int_a^y f(t)\,dt = \int_a^x f(t)\,dt + \int_x^y f(t)\,dt$. It follows that

$$|F(y) - F(x)| = \left|\int_x^y f(t)\,dt\right| \leq \int_x^y |f(t)|\,dt \leq M\,|y - x|.$$

Thus F is a Lipschitz function on $[a, b]$, which implies that F is uniformly continuous on $[a, b]$.

Sets of measure zero

Definition. A subset E of the real line $\mathbb{R}$ is said to have **measure zero** if for any $\varepsilon > 0$ the set E can be covered by countably many open intervals $J_1, J_2, \dots$ such that $\sum_{n=1}^{\infty} |J_n| < \varepsilon$.

Examples. • Any countable set has measure zero.

Indeed, suppose E is a countable set and let $x_1, x_2, \dots$ be a list of all elements of E. Given $\varepsilon > 0$, let

$$J_n = \left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}} \right), \quad n = 1, 2, \dots$$

Then $E \subset J_1 \cup J_2 \cup \dots$ and $|J_n| = \varepsilon/2^n$ for all $n \in \mathbb{N}$ so that $\sum_{n=1}^{\infty} |J_n| = \varepsilon$.

• A nondegenerate interval $[a, b]$ is not a set of measure zero.

• There exist sets of measure zero that are of the same cardinality as $\mathbb{R}$.

Lebesgue's criterion for Riemann integrability

Definition. Suppose $P(x)$ is a property depending on $x \in S$, where $S \subset \mathbb{R}$. We say that $P(x)$ holds for **almost all** $x \in S$ (or **almost everywhere** on S) if the set $\{x \in S \mid P(x) \text{ does not hold}\}$ has measure zero.

Theorem A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on the interval $[a, b]$ if and only if f is bounded on $[a, b]$ and continuous almost everywhere on $[a, b]$.