MATH 409

Advanced Calculus I

Lecture 20:
The fundamental theorem of calculus.
Change of the variable in an integral.

Integral with a variable limit

Suppose $f : [a, b] \to \mathbb{R}$ is an integrable function.

For any $x \in [a, b]$ let $F(x) = \int_a^x f(t)\, dt$

(we assume that $F(a) = 0$).

Theorem 1 The function F is well defined and continuous on $[a, b]$.

Theorem 2 If f is continuous at a point $x \in [a, b]$, then F is differentiable at x and $F'(x) = f(x)$.

Proof of Theorem 2: For any $x, y \in [a, b]$, $x < y$, we have

$$\int_a^y f(t)\,dt = \int_a^x f(t)\,dt + \int_x^y f(t)\,dt.$$

Then

$$F(y) - F(x) - f(x)\,(y - x) = \int_x^y f(t)\,dt - \int_x^y f(x)\,dt$$

so that

$$|F(y) - F(x) - f(x)\,(y - x)| = \left|\int_x^y (f(t) - f(x))\,dt\right|$$
$$\leq \int_x^y |f(t) - f(x)|\,dt \leq \sup_{t\in[x,y]} |f(t) - f(x)|\,(y - x).$$

Finally, $\left|\dfrac{F(y) - F(x)}{y - x} - f(x)\right| \leq \sup_{t\in[x,y]} |f(t) - f(x)|.$

If the function f is right continuous at x, i.e., $f(y) \to f(x)$ as $y \to x+$, then $\sup_{t\in[x,y]} |f(t) - f(x)| \to 0$ as $y \to x+$. It follows that $f(x)$ is the right-hand derivative of F at x. Likewise, one can prove that left continuity of f at x implies that $f(x)$ is the left-hand derivative of F at x.

Fundamental theorem of calculus (part I)

Theorem If a function f is continuous on an interval $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a, b],$$

is continuously differentiable on $[a, b]$. Moreover, $F'(x) = f(x)$ for all $x \in [a, b]$.

Proof: Since f is continuous, it is also integrable on $[a, b]$. As already proved earlier, the integrability of f implies that the function F is well defined and continuous on $[a, b]$. Moreover, $F'(x) = f(x)$ whenever f is continuous at the point x. Therefore the continuity of f on $[a, b]$ implies that $F'(x) = f(x)$ for all $x \in [a, b]$. In particular, F is continuously differentiable on $[a, b]$.

Fundamental theorem of calculus (part II)

Theorem If a function F is differentiable on $[a,b]$ and the derivative F' is integrable on $[a,b]$, then

$$\int_a^x F'(t)\,dt = F(x) - F(a) \quad \text{for all } x \in [a,b].$$

Proof: The case $x = a$ is trivial. Since F' is integrable on $[a,b]$, it is also integrable on any subinterval $[a,x]$, $x \in (a,b)$. Therefore it is no loss to assume that $x = b$.

Consider an arbitrary partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$. Let us choose samples $t_j \in [x_{j-1}, x_j]$ for the Riemann sum $\mathcal{S}(F', P, t_j)$ so that $F(x_j) - F(x_{j-1}) = F'(t_j)\,(x_j - x_{j-1})$ (this is possible due to the Mean Value Theorem). Then $\mathcal{S}(F', P, t_j) = \sum_{j=1}^{n} F'(t_j)\,(x_j - x_{j-1}) = \sum_{j=1}^{n}(F(x_j) - F(x_{j-1}))$ $= F(x_n) - F(x_0) = F(b) - F(a)$. Since the sums $\mathcal{S}(F', P, t_j)$ converge to $\int_a^b F'(t)\,dt$ as $\|P\| \to 0$, the theorem follows.

Indefinite integral

Definition. Given a function $f : [a, b] \to \mathbb{R}$, a function $F : [a, b] \to \mathbb{R}$ is called the **indefinite integral** (or **antiderivative**, or **primitive integral**, or **the primitive**) of f if $F'(x) = f(x)$ for all $x \in [a, b]$. Notation for F: $\int f(x)\,dx$.

If the function f is continuous on $[a, b]$, then the function $F(x) = \int_a^x f(t)\,dt$, $x \in [a, b]$, is an indefinite integral of f due to the Fundamental Theorem of Calculus.

Suppose F is an antiderivative of f. If G is another antiderivative of f, then $G' = F'$ on $[a, b]$. Hence $(G - F)' = G' - F' = 0$ on $[a, b]$. It follows that $G - F$ is a constant function. Conversely, for any constant C the function $G(x) = F(x) + C$ is also an antiderivative of f. Thus the general indefinite integral of f is given by $\int f(x)\,dx = F(x) + C$, where C is an arbitrary constant.

Examples

- $$\int x^\alpha\, dx = \frac{x^{\alpha+1}}{\alpha+1} + C \ \text{ on } \ (0,\infty) \ \text{ for } \ \alpha \neq -1.$$

Indeed, $\left(\frac{x^{\alpha+1}}{\alpha+1}\right)' = \frac{1}{\alpha+1}(x^{\alpha+1})' = \frac{1}{\alpha+1}(\alpha+1)x^\alpha = x^\alpha.$

- $$\int \frac{1}{x}\, dx = \log x + C \ \text{ on } \ (0,\infty).$$

Indeed, $(\log x)' = 1/x$ on $(0,\infty)$.

- $$\int \sin x\, dx = -\cos x + C.$$

- $$\int \cos x\, dx = \sin x + C.$$

Integration by parts

Theorem Suppose that functions f, g are differentiable on $[a, b]$ with the derivatives f', g' integrable on $[a, b]$. Then

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx.$$

Proof: By the Product Rule, $(fg)' = f'g + fg'$ on $[a, b]$. Since the functions f, g, f', g' are integrable on $[a, b]$, so are the products $f'g$ and fg'. Then $(fg)'$ is integrable on $[a, b]$ as well. By the Fundamental Theorem of Calculus,

$$\begin{aligned} f(b)g(b) - f(a)g(a) &= \int_a^b (fg)'(x)\,dx \\ &= \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx. \end{aligned}$$

Corollary Suppose that functions f, g are continuously differentiable on $[a, b]$. Then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \quad \text{on } [a, b].$$

To simplify notation, it is convenient to use the **Leibniz differential** df of a function f defined by $df(x) = f'(x)\,dx = \frac{df}{dx}\,dx$. Another convenient notation is $f(x)\big|_{x=a}^{b}$ or simply $f(x)\big|_{a}^{b}$, which denotes the difference $f(b) - f(a)$.

Now the formula of integration by parts can be rewritten as

$$\int_a^b f(x)\,dg(x) = f(x)g(x)\Big|_a^b - \int_a^b g(x)\,df(x)$$

for definite integrals and as

$$\int f\,dg = fg - \int g\,df$$

for indefinite integrals.

Examples

- $$\int \log x \, dx = x \log x - x + C \quad \text{on} \quad (0, \infty).$$

Integrating by parts, we obtain

$$\int \log x \, dx = x \log x - \int x \, d(\log x) = x \log x$$
$$- \int x (\log x)' \, dx = x \log x - \int 1 \, dx = x \log x - x + C.$$

- $$\int_0^{\pi/2} x \sin x \, dx = 1.$$

Integrating by parts, we obtain

$$\int_0^{\pi/2} x \sin x \, dx = -x \cos x \Big|_0^{\pi/2} - \int_0^{\pi/2} (-\cos x) \, dx = \sin x \Big|_0^{\pi/2} = 1.$$

Change of the variable in an integral

Theorem If ϕ is continuously differentiable on a closed, nondegenerate interval $[a, b]$ and f is continuous on $\phi([a, b])$, then

$$\int_{\phi(a)}^{\phi(b)} f(t)\,dt = \int_a^b f(\phi(x))\,\phi'(x)\,dx = \int_a^b f(\phi(x))\,d\phi(x).$$

Remarks. • It is possible that $\phi(a) \geq \phi(b)$. To make sense of the integral in this case, we set

$$\int_c^d f(t)\,dt = -\int_d^c f(t)\,dt$$

if $c > d$. Also, we set the integral to be 0 if $c = d$.

• $t = \phi(x)$ is a proper change of the variable only if the function ϕ is strictly monotone. However the theorem holds even without this assumption.

Proof of the theorem: Let us define two functions:

$$F(u) = \int_{\phi(a)}^{u} f(t)\,dt, \quad u \in \phi([a,b]);$$

and

$$G(x) = \int_{a}^{x} f(\phi(s))\,\phi'(s)\,ds, \quad x \in [a,b].$$

It follows from the Fundamental Theorem of Calculus that $F'(u) = f(u)$ and $G'(x) = f(\phi(x))\,\phi'(x)$. By the Chain Rule,

$$(F \circ \phi)'(x) = F'(\phi(x))\phi'(x) = f(\phi(x))\phi'(x) = G'(x).$$

Therefore $(F(\phi(x)) - G(x))' = 0$ for all $x \in [a,b]$. It follows that the function $F(\phi(x)) - G(x)$ is constant on $[a,b]$. In particular, $F(\phi(b)) - G(b) = F(\phi(a)) - G(a) = 0 - 0 = 0$.