MATH 409

Advanced Calculus I

Lecture 21:
Review for Test 2.

Topics for Test 2

- Derivative of a function
- Differentiability theorems
- Derivative of the inverse function
- The mean value theorem
- Taylor's formula
- l'Hôpital's rule

- Darboux sums, Riemann sums, the Riemann integral
- Properties of integrals
- The fundamental theorem of calculus
- Integration by parts
- Change of the variable in an integral

Wade's book: 4.1–4.5, 5.1–5.3

Differentiability theorems

Theorem If functions f and g are differentiable at a point $a \in \mathbb{R}$, then their sum $f+g$, difference $f-g$, and product $f \cdot g$ are also differentiable at a. Moreover,

$$(f+g)'(a) = f'(a) + g'(a),$$
$$(f-g)'(a) = f'(a) - g'(a),$$
$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a).$$

If, additionally, $g(a) \neq 0$ then the quotient f/g is also differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

Theorem If a function f is differentiable at a point $a \in \mathbb{R}$ and a function g is differentiable at $f(a)$, then the composition $g \circ f$ is differentiable at a. Moreover,

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

More theorems to know

Theorem If a function f is differentiable at a point c, then it is continuous at c.

Rolle's Theorem If a function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

Mean Value Theorem If a function f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)\,(b - a)$.

Theorem Suppose that a function f is continuous on $[a, b]$ and differentiable on (a, b). Then the following hold.

(i) f is increasing on $[a, b]$ if and only if $f' \geq 0$ on (a, b).
(ii) f is decreasing on $[a, b]$ if and only if $f' \leq 0$ on (a, b).
(iii) f is constant on $[a, b]$ if and only if $f' = 0$ on (a, b).

Properties of integrals

Theorem If functions f, g are integrable on an interval $[a, b]$, then the sum $f + g$ is also integrable on $[a, b]$ and

$$\int_a^b \big(f(x) + g(x)\big)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

Theorem If a function f is integrable on $[a, b]$, then for each $\alpha \in \mathbb{R}$ the scalar multiple αf is also integrable on $[a, b]$ and

$$\int_a^b \alpha f(x)\, dx = \alpha \int_a^b f(x)\, dx.$$

Properties of integrals

Theorem If a function f is integrable on $[a, b]$ then for any $c \in (a, b)$,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Theorem If functions f, g are integrable on $[a, b]$ and $f(x) \le g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Fundamental theorem of calculus

Theorem If a function f is continuous on an interval $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a, b],$$

is continuously differentiable on $[a, b]$. Moreover, $F'(x) = f(x)$ for all $x \in [a, b]$.

Theorem If a function F is differentiable on $[a, b]$ and the derivative F' is integrable on $[a, b]$, then

$$\int_a^x F'(t)\,dt = F(x) - F(a) \quad \text{for all } x \in [a, b].$$

Sample problems for Test 2

Problem 1 (20 pts.) Prove the Chain Rule: if a function f is differentiable at a point c and a function g is differentiable at $f(c)$, then the composition $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Problem 2 (25 pts.) Find the following limits of functions:

(i) $\lim\limits_{x\to 0} (1+x)^{1/x}$, **(ii)** $\lim\limits_{x\to +\infty} (1+x)^{1/x}$,

(iii) $\lim\limits_{x\to 0+} x^x$.

Sample problems for Test 2

Problem 3 (20 pts.) Find the limit of a sequence

$$x_n = \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}}, \quad n = 1, 2, \ldots,$$

where k is a natural number.

Sample problems for Test 2

Problem 4 (25 pts.) Find indefinite integrals and evaluate definite integrals:

(i) $\displaystyle\int \frac{x^2}{1-x}\,dx$, **(ii)** $\displaystyle\int_0^{\pi} \sin^2(2x)\,dx$,

(iii) $\displaystyle\int \log^3 x\,dx$, **(iv)** $\displaystyle\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$,

(v) $\displaystyle\int_0^{1} \frac{1}{\sqrt{4-x^2}}\,dx$.

Sample problems for Test 2

Bonus Problem 5 (15 pts.) Suppose that a function $p:\mathbb{R}\to\mathbb{R}$ is locally a polynomial, which means that for every $c\in\mathbb{R}$ there exists $\varepsilon>0$ such that p coincides with a polynomial on the interval $(c-\varepsilon, c+\varepsilon)$. Prove that p is a polynomial.

Bonus Problem 6 (15 pts.) Show that a function

$$f(x)=\begin{cases}\exp\left(-\dfrac{1}{1-x^2}\right) & \text{if } |x|<1,\\ 0 \text{ if } |x|\geq 1\end{cases}$$

is infinitely differentiable on $\mathbb{R}$.

Problem 2 Find the following limits of functions:

(i) $\lim_{x\to 0}(1+x)^{1/x}$, **(ii)** $\lim_{x\to+\infty}(1+x)^{1/x}$.

The function $f(x)=(1+x)^{1/x}$ is well defined on $(-1,0)\cup(0,\infty)$. Since $f(x)>0$ for all $x>-1$, $x\neq 0$, a function $g(x)=\log f(x)$ is well defined on $(-1,0)\cup(0,\infty)$ as well. For any $x>-1$, $x\neq 0$, we have $g(x)=\log(1+x)^{1/x}=x^{-1}\log(1+x)$. Hence $g=h_1/h_2$, where the functions $h_1(x)=\log(1+x)$ and $h_2(x)=x$ are continuously differentiable on $(-1,\infty)$. Since $h_1(0)=h_2(0)=0$, it follows that $\lim_{x\to 0}h_1(x)=\lim_{x\to 0}h_2(x)=0$. By l'Hôpital's Rule,

$$\lim_{x\to 0}\frac{h_1(x)}{h_2(x)}=\lim_{x\to 0}\frac{h_1'(x)}{h_2'(x)}$$

assuming the latter limit exists.

Since $h_1'(0) = (1+x)^{-1}|_{x=0} = 1$ and $h_2'(0) = 1$, we obtain

$$\lim_{x\to 0} \frac{h_1(x)}{h_2(x)} = \lim_{x\to 0} \frac{h_1'(x)}{h_2'(x)} = \frac{\lim\limits_{x\to 0} h_1'(x)}{\lim\limits_{x\to 0} h_2'(x)} = \frac{1}{1} = 1.$$

Further, $\lim\limits_{x\to+\infty} h_1(x) = \lim\limits_{x\to+\infty} h_2(x) = +\infty$. At the same time, $h_1'(x) = (1+x)^{-1} \to 0$ as $x \to +\infty$ while h_2' is identically 1. Using l'Hôpital's Rule and a limit theorem, we obtain

$$\lim_{x\to+\infty} \frac{h_1(x)}{h_2(x)} = \lim_{x\to+\infty} \frac{h_1'(x)}{h_2'(x)} = \frac{\lim\limits_{x\to+\infty} h_1'(x)}{\lim\limits_{x\to+\infty} h_2'(x)} = \frac{0}{1} = 0.$$

Since $f = e^g$, a composition of g with a continuous function, it follows that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{g(x)} = \exp\left(\lim_{x\to 0} g(x)\right) = e^1 = e,$$

$$\lim_{x\to+\infty} f(x) = \exp\Bigl(\lim_{x\to+\infty} g(x)\Bigr) = e^0 = 1.$$

Problem 3 Find the limit of a sequence

$$x_n = \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}}, \quad n = 1, 2, \dots,$$

where k is a natural number.

The general element of the sequence can be represented as

$$x_n = \frac{1^k + 2^k + \cdots + n^k}{n^k} \cdot \frac{1}{n} = \left(\frac{1}{n}\right)^k \frac{1}{n} + \left(\frac{2}{n}\right)^k \frac{1}{n} + \cdots + \left(\frac{n}{n}\right)^k \frac{1}{n},$$

which shows that x_n is a Riemann sum of the function $f(x) = x^k$ on the interval $[0, 1]$ that corresponds to the partition $P_n = \{0, 1/n, 2/n, \dots, (n-1)/n, 1\}$ and samples $t_j = j/n$, $j = 1, 2, \dots, n$. The norm of the partition is $\|P_n\| = 1/n$. Since $\|P_n\| \to 0$ as $n \to \infty$ and the function f is integrable on $[0, 1]$, the Riemann sums x_n converge to the integral:

$$\lim_{n\to\infty} x_n = \int_0^1 x^k\, dx = \left.\frac{x^{k+1}}{k+1}\right|_{x=0}^{1} = \frac{1}{k+1}.$$

Problem 4 Find indefinite integrals and evaluate definite integrals:

(i) $\displaystyle\int \frac{x^2}{1-x}\,dx.$

To find the indefinite integral of this rational function, we expand it into the sum of a polynomial and a simple fraction:

$$\frac{x^2}{1-x} = \frac{x^2-1+1}{1-x} = \frac{x^2-1}{1-x} + \frac{1}{1-x} = -x-1-\frac{1}{x-1}.$$

Since the domain of the function is $(-\infty,1)\cup(1,\infty)$, the indefinite integral has different representations on the intervals $(-\infty,1)$ and $(1,\infty)$:

$$\int \frac{x^2}{1-x}\,dx = \begin{cases} -x^2/2 - x - \log(1-x) + C_1, & x<1, \\ -x^2/2 - x - \log(x-1) + C_2, & x>1. \end{cases}$$

Problem 4 Find indefinite integrals and evaluate definite integrals:

(ii) $\int_0^{\pi} \sin^2(2x)\, dx$.

To integrate this function, we use a trigonometric formula $1 - \cos(2\alpha) = 2\sin^2\alpha$ and a new variable $u = 4x$:

$$\int_0^{\pi} \sin^2(2x)\, dx = \int_0^{\pi} \frac{1-\cos(4x)}{2}\, dx$$

$$= \int_0^{\pi} \frac{1-\cos(4x)}{8}\, d(4x) = \int_0^{4\pi} \frac{1-\cos u}{8}\, du$$

$$= \left.\frac{u - \sin u}{8}\right|_{u=0}^{4\pi} = \frac{\pi}{2}.$$

Problem 4 Find indefinite integrals and evaluate definite integrals:

(iii) $\int \log^3 x\,dx$.

To find this indefinite integral, we integrate by parts:

$$\int \log^3 x\,dx = x\log^3 x - \int x\,d(\log^3 x) = x\log^3 x - \int x(\log^3 x)'\,dx$$

$$= x\log^3 x - \int 3\log^2 x\,dx = x\log^3 x - 3x\log^2 x + \int x\,d(3\log^2 x)$$

$$= x\log^3 x - 3x\log^2 x + \int 6\log x\,dx$$

$$= x\log^3 x - 3x\log^2 x + 6x\log x - \int x\,d(6\log x)$$

$$= x\log^3 x - 3x\log^2 x + 6x\log x - \int 6\,dx$$

$$= x\log^3 x - 3x\log^2 x + 6x\log x - 6x + C.$$

Problem 4 Find indefinite integrals and evaluate definite integrals:

(iv) $\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\, dx.$

To integrate this function, we introduce a new variable $u = 1 - x^2$:

$$\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\, dx = -\frac{1}{2}\int_0^{1/2} \frac{(1-x^2)'}{\sqrt{1-x^2}}\, dx$$

$$= -\frac{1}{2}\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\, d(1-x^2) = -\frac{1}{2}\int_1^{3/4} \frac{1}{\sqrt{u}}\, du$$

$$= \int_{3/4}^{1} \frac{1}{2\sqrt{u}}\, du = \sqrt{u}\,\Big|_{u=3/4}^{1} = 1 - \frac{\sqrt{3}}{2}.$$

Problem 4 Find indefinite integrals and evaluate definite integrals:

(v) $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\, dx.$

To integrate this function, we use a substitution $x = 2\sin t$ (observe that x changes from 0 to 1 when t changes from 0 to $\pi/6$):

$$\int_0^1 \frac{1}{\sqrt{4-x^2}}\, dx = \int_0^{\pi/6} \frac{1}{\sqrt{4-(2\sin t)^2}}\, d(2\sin t)$$

$$= \int_0^{\pi/6} \frac{(2\sin t)'}{\sqrt{4-4\sin^2 t}}\, dt = \int_0^{\pi/6} \frac{2\cos t}{\sqrt{4\cos^2 t}}\, dt$$

$$= \int_0^{\pi/6} \frac{2\cos t}{2\cos t}\, dt = \int_0^{\pi/6} 1\, dx = \frac{\pi}{6}.$$

Bonus Problem 5 Suppose that a function $p : \mathbb{R} \to \mathbb{R}$ is locally a polynomial, which means that for every $c \in \mathbb{R}$ there exists $\varepsilon > 0$ such that p coincides with a polynomial on the interval $(c - \varepsilon, c + \varepsilon)$. Prove that p is a polynomial.

For any $c \in \mathbb{R}$ let p_c denote a polynomial and ε_c denote a positive number such that $p(x) = p_c(x)$ for all $x \in (c - \varepsilon_c, c + \varepsilon_c)$. Consider two sets

$$E_+ = \{x > 0 \mid p(x) \neq p_0(x)\} \text{ and } E_- = \{x < 0 \mid p(x) \neq p_0(x)\}.$$

We are going to show that $E_+ = E_- = \emptyset$. This would imply that $p = p_0$ on the entire real line.

Assume that the set E_+ is not empty. Clearly, E_+ is bounded below. Hence $d = \inf E_+$ is a well-defined real number. Note that $E_+ \subset [\varepsilon_0, \infty)$. Therefore $d \geq \varepsilon_0 > 0$.

Observe that $p(x) = p_0(x)$ for $x \in (0, d)$ and $p(x) = p_d(x)$ for $x \in (d - \varepsilon_d, d + \varepsilon_d)$. The interval $(0, d)$ overlaps with the interval $(d - \varepsilon_d, d + \varepsilon_d)$. Hence p_d coincides with p_0 on the intersection $(0, d) \cap (d - \varepsilon_d, d + \varepsilon_d)$. Equivalently, the difference $p_d - p_0$ is zero on $(0, d) \cap (d - \varepsilon_d, d + \varepsilon_d)$. Since $p_d - p_0$ is a polynomial and any nonzero polynomial has only finitely many roots, we conclude that $p_d - p_0$ is identically 0. Then the polynomials p_d and p_0 are the same. It follows that $p(x) = p_0(x)$ for $x \in (0, d + \varepsilon_d)$ so that $d \neq \inf E_+$, a contradiction. Thus $E_+ = \emptyset$. Similarly, we prove that the set E_- is empty as well. Since $E_+ = E_- = \emptyset$, the function p coincides with the polynomial p_0.