MATH 409

Advanced Calculus I

Lecture 22: Improper Riemann integrals.

Improper Riemann integral

If a function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$, then the function $F(x) = \int_a^x f(t)\,dt$ is well defined and continuous on $[a, b]$. In particular, $F(c) \to F(b)$ as $c \to b-$, i.e.,

$$\int_a^b f(x)\,dx = \lim_{c \to b+} \int_a^c f(x)\,dx.$$

Now suppose that f is defined on the semi-open interval $J = [a, b)$ and is integrable on any closed interval $[c, d] \subset J$ (such a function is called **locally integrable** on J). Then all integrals in the right-hand side are well defined and the limit might exist even if f is not integrable on $[a, b]$.

If this is the case, then f is called **improperly integrable** on J and the limit is called the **(improper) integral** of f on $[a, b)$.

Similarly, one defines improper integrability on the semi-open interval $(a, b]$.

Suppose a function f is locally integrable on a semi-open interval $J = [a, b)$ or $(a, b]$. Then there are two possible obstructions for f to be integrable on $[a, b]$: **(i)** the function f is not bounded on J, and **(ii)** the interval J is not bounded.

Examples.

- Function $f(x) = 1/\sqrt{x}$ is improperly integrable on $(0, 1]$.

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0+} \int_c^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0+} 2\sqrt{x}\Big|_{x=c}^{1}$$
$$= \lim_{c\to 0+} (2 - 2\sqrt{c}) = 2.$$

- Function $g(x) = x^{-2}$ is improperly integrable on $[1, \infty)$.

$$\int_1^\infty x^{-2}\,dx = \lim_{c\to +\infty} \int_1^c x^{-2}\,dx = \lim_{c\to +\infty} -x^{-1}\Big|_{x=1}^{c}$$
$$= \lim_{c\to +\infty} (1 - c^{-1}) = 1.$$

Properties of improper integrals

Since an improper Riemann integral is a limit of proper integrals, the properties of improper integrals are analogous to those of proper integrals (and derived using limit theorems).

Theorem Let $f : [a, b) \to \mathbb{R}$ be a function integrable on any closed interval $[a_1, b_1] \subset [a, b)$. Given $c \in (a, b)$, the function f is improperly integrable on $[c, b)$ if and only if it is improperly integrable on $[a, b)$. In the case of integrability,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Sketch of the proof: For any $d \in (c, b)$ we have the following equality involving proper Riemann integrals:

$$\int_a^d f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx.$$

The theorem is proved by taking the limit as $d \to b-$.

Theorem Suppose that a function $f : (a, b) \to \mathbb{R}$ is integrable on any closed interval $[c, d] \subset (a, b)$. Given a number $I \in \mathbb{R}$, the following conditions are equivalent:

(i) for some $c \in (a, b)$ the function f is improperly integrable on $(a, c]$ and $[c, b)$, and $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = I$;

(ii) for every $c \in (a, b)$ the function f is improperly integrable on $(a, c]$ and $[c, b)$, and $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = I$;

(iii) for every $c \in (a, b)$ the function f is improperly integrable on $(a, c]$ and $\int_a^c f(x)\,dx \to I$ as $c \to b-$;

(iv) for every $c \in (a, b)$ the function f is improperly integrable on $[c, b)$ and $\int_c^b f(x)\,dx \to I$ as $c \to a+$.

Improper integral: two singular points

Definition. A function $f : (a, b) \to \mathbb{R}$ is called **improperly integrable** on the open interval (a, b) if for some (and then for any) $c \in (a, b)$ it is improperly integrable on semi-open intervals $(a, c]$ and $[c, b)$. The **integral** of f is defined by

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In view of the previous theorem, the integral does not depend on c. It can also be computed as a repeated limit:

$$\int_a^b f(x)\,dx = \lim_{d\to b-}\left(\lim_{c\to a+}\int_c^d f(x)\,dx\right) = \lim_{c\to a+}\left(\lim_{d\to b-}\int_c^d f(x)\,dx\right).$$

Finally, the integral can be computed as a double limit (i.e., the limit of a function of two variables):

$$\int_a^b f(x)\,dx = \lim_{\substack{c\to a+\\ d\to b-}}\int_c^d f(x)\,dx.$$

More properties of improper integrals

- If a function f is integrable on a closed interval $[a, b]$ or improperly integrable on one of the semi-open intervals $[a, b)$ and $(a, b]$, then it is also improperly integrable on the open interval (a, b) with the same value of the integral.

- If functions f, g are improperly integrable on (a, b), then for any $\alpha, \beta \in \mathbb{R}$ the linear combination $\alpha f + \beta g$ is also improperly integrable on (a, b) and

$$\int_a^b \big(\alpha f(x) + \beta g(x)\big)\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx.$$

- Suppose a function $f : (a, b) \to \mathbb{R}$ is locally integrable and has an antiderivative F. Then f is improperly integrable on (a, b) if and only if $F(x)$ has finite limits as $x \to a+$ and as $x \to b-$, in which case

$$\int_a^b f(x)\, dx = \lim_{x \to b-} F(x) - \lim_{x \to a+} F(x).$$

Comparison Theorems for improper integrals

Theorem 1 Suppose that functions f, g are improperly integrable on (a, b). If $f(x) \leq g(x)$ for all $x \in (a, b)$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Theorem 2 Suppose that functions f, g are locally integrable on (a, b). If the function g is improperly integrable on (a, b) and $0 \leq f(x) \leq g(x)$ for all $x \in (a, b)$, then f is also improperly integrable on (a, b).

Theorem 3 Suppose that functions f, g, h are locally integrable on (a, b). If the functions g, h are improperly integrable on (a, b) and $h(x) \leq f(x) \leq g(x)$ for all $x \in (a, b)$, then f is also improperly integrable on (a, b) and

$$\int_a^b h(x)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Examples

- Function $f(x) = x^{-2}$ is not improperly integrable on $(0, \infty)$.

Indeed, the antiderivative of the function f, which is $F(x) = -x^{-1}$, has a finite limit as $x \to +\infty$ but diverges to infinity as $x \to 0+$.

- Function $g(x) = x^{-2} \cos x$ is improperly integrable on $[1, \infty)$.

We have $-f(x) \leq g(x) \leq f(x) = x^{-2}$ for all $x \geq 1$. Since the function f is improperly integrable on $[1, \infty)$, it follows that $-f$ is also improperly integrable on $[1, \infty)$. By the Comparison Theorem for improper integrals, the function g is improperly integrable on $[1, \infty)$ as well.

Examples

- Function $f(x) = e^{-x}$ is improperly integrable on $[0, \infty)$.

Indeed, the antiderivative of the function f, which is $F(x) = -e^{-x}$, has a finite limit as $x \to +\infty$.

- Function $g(x) = e^{-x^2}$ is improperly integrable on $(-\infty, \infty)$.

We have $0 \leq g(x) \leq f(x) = e^{-x}$ for all $x \geq 1$. Since the function f is improperly integrable on $[0, \infty)$, it follows that g is improperly integrable on $[1, \infty)$. Since the function g is even, $g(-x) = g(x)$, it follows that g is also improperly integrable on $(-\infty, -1]$. Finally, g is properly integrable on $[-1, 1]$.

Examples

- Function $f(x) = x^{-1}\sin x$ is improperly integrable on $[1, \infty)$.

To show improper integrability, we integrate by parts:

$$\begin{aligned}\int_1^c x^{-1}\sin x\,dx &= -\int_1^c x^{-1}\,d(\cos x)\\ &= -\,x^{-1}\cos x\Big|_{x=1}^{c} + \int_1^c \cos x\,d(x^{-1})\\ &= \cos 1 - c^{-1}\cos c - \int_1^c x^{-2}\cos x\,dx.\end{aligned}$$

Since the function $g(x) = x^{-2}\cos x$ is improperly integrable on $[1, \infty)$ and $c^{-1}\cos c \to 0$ as $c \to +\infty$, it follows that f is improperly integrable on $[1, \infty)$.

Absolute integrability

Definition. A function $f : (a, b) \to \mathbb{R}$ is called **absolutely integrable** on (a, b) if f is locally integrable on (a, b) and the function $|f|$ is improperly integrable on (a, b).

Theorem If a function f is absolutely integrable on (a, b), then it is also improperly integrable on (a, b) and

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Proof: Since $|f|$ is improperly integrable on (a, b), so is $-|f|$. Clearly, $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in (a, b)$. By the Comparison Theorems for improper integrals, the function f is improperly integrable on (a, b) and

$$-\int_a^b |f(x)|\, dx \leq \int_a^b f(x)\, dx \leq \int_a^b |f(x)|\, dx.$$

Examples

- For any nonnegative function, the absolute integrability is equivalent to improper integrability.

In particular, the function $f_1(x) = x^{-2}$ is absolutely integrable on $[1, \infty)$ and is not on $(0, \infty)$. The function $f_2(x) = 1/\sqrt{x}$ is absolutely integrable on $(0, 1)$. The function $f_3(x) = e^{-x^2}$ is absolutely integrable on $(-\infty, \infty)$.

- Function $f(x) = e^{-x^2} \sin x$ is absolutely integrable on $(-\infty, \infty)$.

Indeed, the function f is locally integrable on $(-\infty, \infty)$, a function $g(x) = e^{-x^2}$ is improperly integrable on $(-\infty, \infty)$, and $|f(x)| \leq g(x)$ for all $x \in \mathbb{R}$.

Counterexamples

- Function $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is not absolutely integrable on $(0,1)$.

Indeed, the function f is not locally integrable on $(0,1)$. At the same time, the function $|f|$ is constant and hence (properly) integrable on $(0,1)$.

- Function $f(x) = x^{-1} \sin x$ is not absolutely integrable on $[1,\infty)$.

For any $n \in \mathbb{N}$, $\displaystyle\int_{n\pi}^{(n+1)\pi} |f(x)|\,dx \geq \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{(n+1)\pi}\,dx$

$$= \frac{1}{(n+1)\pi} \int_0^{\pi} \sin x\,dx = \frac{2}{(n+1)\pi} \geq \frac{1}{n\pi} \geq \frac{1}{\pi} \int_{n\pi}^{(n+1)\pi} \frac{dx}{x}.$$

It remains to notice that $g(x) = 1/x$ is not improperly integrable on $[\pi,\infty)$.