MATH 409–501/503 Fall 2013

Sample problems for Test 1: Solutions

Any problem may be altered or replaced by a different one!

Problem 1 (15 pts.) Prove that for any $n \in \mathbb{N}$,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

The proof is by induction on n. First we consider the case $n = 1$. In this case the formula reduces to $1^3 = 1^2 \cdot 2^2/4$, which is a true equality. Now assume that the formula holds for $n = k$, that is,

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}.$$

Adding $(k+1)^3$ to both sides of this equality, we get

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 = (k+1)^2\left(\frac{k^2}{4} + (k+1)\right)$$

$$= (k+1)^2\,\frac{k^2+4k+4}{4} = \frac{(k+1)^2(k+2)^2}{4},$$

which means that the formula holds for $n = k+1$ as well. By induction, the formula holds for every natural number n.

Problem 2 (30 pts.) Let $\{F_n\}$ be the sequence of Fibonacci numbers: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

(i) Show that the sequence $\{F_{2k}/F_{2k-1}\}_{k\in\mathbb{N}}$ is increasing while the sequence $\{F_{2k+1}/F_{2k}\}_{k\in\mathbb{N}}$ is decreasing.

Let $x_n = F_{n+1}/F_n$, $n \in \mathbb{N}$. Then

$$x_{n+1} = \frac{F_{n+2}}{F_{n+1}} = \frac{F_n + F_{n+1}}{F_{n+1}} = 1 + \frac{F_n}{F_{n+1}} = 1 + \frac{1}{x_n}$$

for all $n \in \mathbb{N}$. We obtain that $x_1 = 1$, $x_2 = 1 + 1/x_1 = 2$, $x_3 = 1 + 1/x_2 = 3/2$, $x_4 = 1 + 1/x_3 = 5/3$. Note that the function $f(x) = 1 + 1/x$ is strictly decreasing on the interval $(0,\infty)$ and maps this interval to itself. Therefore its second iteration $g = f \circ f$ is strictly increasing on $(0,\infty)$ and also maps this interval to itself. Since $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$, it follows that $x_{n+2} = g(x_n)$ for all $n \in \mathbb{N}$. The above computations show that $x_1 < x_3 < x_4 < x_2$. Since g is strictly increasing, it follows by induction on k that $x_{2k-1} < x_{2k+1} < x_{2k+2} < x_{2k}$. In particular, the sequence $\{x_{2k-1}\}$ is strictly increasing while the sequence $\{x_{2k}\}$ is strictly decreasing.

(ii) Prove that $\lim_{n\to\infty} \dfrac{F_{n+1}}{F_n} = \dfrac{\sqrt{5}+1}{2}$.

By the above the sequence $\{x_{2k-1}\}$ is strictly increasing while the sequence $\{x_{2k}\}$ is strictly decreasing. Moreover, $x_{2k-1} < x_{2k}$ for all $k \in \mathbb{N}$, which implies that both sequences are bounded. It follows that both sequences are converging to positive limits c_1 and c_2, respectively. To prove that $\lim_{n\to\infty} F_{n+1}/F_n = (\sqrt{5}+1)/2$, it is enough to show that $c_1 = c_2 = (\sqrt{5}+1)/2$. For any $x > 0$ we obtain

$$g(x) = f(f(x)) = f\left(1 + \frac{1}{x}\right) = 1 + \frac{1}{1+\frac{1}{x}} = 1 + \frac{1}{\frac{x+1}{x}} = 1 + \frac{x}{x+1} = \frac{2x+1}{x+1}.$$

It follows that $g(x_{2k-1}) \to g(c_1)$ and $g(x_{2k}) \to g(c_2)$ as $k \to \infty$. However $g(x_{2k-1}) = x_{2k+1}$ and $g(x_{2k}) = x_{2k+2}$, which implies that $g(c_1) = c_1$ and $g(c_2) = c_2$. Since

$$x - g(x) = \frac{x(x+1)}{x+1} - \frac{2x+1}{x+1} = \frac{x^2 - x - 1}{x+1},$$

c_1 and c_2 are roots of the equation $x^2 - x - 1 = 0$. This equation has two roots, $(1-\sqrt{5})/2$ and $(\sqrt{5}+1)/2$. One of the roots is negative. Thus both c_1 and c_2 are equal to the other root, $(\sqrt{5}+1)/2$.

Problem 3 (25 pts.) Prove the Extreme Value Theorem: if $f : [a, b] \to \mathbb{R}$ is a continuous function on a closed bounded interval $[a, b]$, then f is bounded and attains its extreme values (maximum and minimum) on $[a, b]$.

First let us prove that the function f is bounded. Assume the contrary. Then for every $n \in \mathbb{N}$ there exists a point $x_n \in [a, b]$ such that $|f(x_n)| > n$. We obtain a sequence $\{x_n\}$ of elements of $[a, b]$ such that the sequence $\{f(x_n)\}$ diverges to infinity. Since the sequence $\{x_n\}$ is bounded, it has a convergent subsequence $\{x_{n_k}\}$ due to the Bolzano-Weierstrass Theorem. Let c be the limit of x_{n_k} as $k \to \infty$. Since $a \le x_{n_k} \le b$ for all k, the Comparison Theorem implies that $a \le c \le b$, i.e., $c \in [a, b]$. Then the function f is continuous at c. As a consequence, $f(x_{n_k}) \to f(c)$ as $k \to \infty$. However the sequence $\{f(x_{n_k})\}$ is a subsequence of $\{f(x_n)\}$ and hence diverges to infinity. This contradiction shows that the assumption was wrong: the function f is bounded.

Since the function f is bounded, the image $f([a, b])$ is a bounded subset of $\mathbb{R}$. Let $m = \inf f([a, b])$, $M = \sup f([a, b])$. For any $n \in \mathbb{N}$ the number $M - 1/n$ is not an upper bound of the set $f([a, b])$ while $m + 1/n$ is not a lower bound of $f([a, b])$. Hence we can find points $y_n, z_n \in [a, b]$ such that $f(y_n) > M - 1/n$ and $f(z_n) < m + 1/n$. At the same time, $m \le f(x) \le M$ for all $x \in [a, b]$. It follows that $f(y_n) \to M$ and $f(z_n) \to m$ as $n \to \infty$. By the Bolzano-Weierstrass Theorem, the sequence $\{y_n\}$ has a subsequence $\{y_{n_k}\}$ converging to some c_1. The sequence $\{z_n\}$ also has a subsequence $\{z_{m_k}\}$ converging to some c_2. Moreover, $c_1, c_2 \in [a, b]$. The continuity of f implies that $f(y_{n_k}) \to f(c_1)$ and $f(z_{m_k}) \to f(c_2)$ as $k \to \infty$. Since $\{f(y_{n_k})\}$ is a subsequence of $\{f(y_n)\}$ and $\{f(z_{m_k})\}$ is a subsequence of $\{f(z_n)\}$, we conclude that $f(c_1) = M$ and $f(c_2) = m$. Thus the function f attains its maximum M on the interval $[a, b]$ at the point c_1 and its minimum m at the point c_2.

Problem 4 (20 pts.) Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(-1) = f(0) = f(1) = 0$ and $f(x) = \dfrac{x-1}{x^2-1} \sin \dfrac{1}{x}$ for $x \in \mathbb{R} \setminus \{-1, 0, 1\}$.

(i) Determine all points at which the function f is continuous.

The polynomial functions $g_1(x) = x - 1$ and $g_2(x) = x^2 - 1$ are continuous on the entire real line. Moreover, $g_2(x) = 0$ if and only if $x = 1$ or -1. Therefore the quotient $g(x) = g_1(x)/g_2(x)$ is well defined and continuous on $\mathbb{R} \setminus \{-1, 1\}$. Further, the function $h_1(x) = 1/x$ is continuous on $\mathbb{R} \setminus \{0\}$. Since the function $h_2(x) = \sin x$ is continuous on $\mathbb{R}$, the composition function $h(x) = h_2(h_1(x))$ is continuous on $\mathbb{R} \setminus \{0\}$. Clearly, $f(x) = g(x)h(x)$ for all $x \in \mathbb{R} \setminus \{-1, 0, 1\}$. It follows that the function f is continuous on $\mathbb{R} \setminus \{-1, 0, 1\}$.

It remains to determine whether the function f is continuous at points -1, 0, and 1. Observe that $g(x) = 1/(x + 1)$ for all $x \in \mathbb{R} \setminus \{-1, 1\}$. Therefore $g(x) \to 1$ as $x \to 0$, $g(x) \to 1/2$ as $x \to 1$, and $g(x) \to \pm\infty$ as $x \to -1$. Since the function h is continuous at -1 and 1, we have $h(x) \to h(-1) = -\sin 1$ as $x \to -1$ and $h(x) \to h(1) = \sin 1$ as $x \to 1$. Note that $\sin 1 \neq 0$ since $0 < 1 < \pi/2$. It follows that $f(x) \to \pm\infty$ as $x \to -1$. In particular, f is discontinuous at -1. Also, $f(x) \to \frac{1}{2} \sin 1$ as $x \to 1$. Since $f(1) = 0$, the function f has a removable discontinuity at 1. Finally, the function f is not continuous at 0 since it has no limit at 0. To be precise, let $x_n = (\pi/2 + 2\pi n)^{-1}$ and $y_n = (-\pi/2 + 2\pi n)^{-1}$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ and $\{y_n\}$ are two sequences of positive numbers converging to 0. We have $h(x_n) = 1$ and $h(y_n) = -1$ for all $n \in \mathbb{N}$. It follows that $f(x_n) \to 1$ and $f(y_n) \to -1$ as $n \to \infty$. Hence there is no limit of $f(x)$ as $x \to 0+$.

(ii) Is the function f uniformly continuous on the interval $(0, 1)$? Is it uniformly continuous on the interval $(1, 2)$? Explain.

Any function uniformly continuous on the open interval $(0, 1)$ can be extended to a continuous function on $[0, 1]$. As a consequence, such a function has a right-hand limit at 0. However it was shown above that the function f has no right-hand limit at 0. Therefore f is not uniformly continuous on $(0, 1)$.

The function f is continuous on $(1, 2]$ and has a removable singularity at 1. Changing the value of f at 1 to the limit at 1, we obtain a function continuous on $[1, 2]$. We know that every function continuous on the closed interval $[1, 2]$ is also uniformly continuous on $[1, 2]$. Further, any function uniformly continuous on the set $[1, 2]$ is also uniformly continuous on its subset $(1, 2)$. Since the redefined function coincides with f on $(1, 2)$, we conclude that f is uniformly continuous on $(1, 2)$.

Bonus Problem 5 (15 pts.) Given a set X, let $\mathcal{P}(X)$ denote the set of all subsets of X. Prove that $\mathcal{P}(X)$ is not of the same cardinality as X.

We have to prove that there is no bijective map of X onto $\mathcal{P}(X)$. Let us consider an arbitrary map $f : X \to \mathcal{P}(X)$. The image $f(x)$ of an element $x \in X$ under this map is a subset of X. Let $E = \{x \in X \mid x \notin f(x)\}$. By definition of the set E, any element $x \in X$ belongs to E if and only if it does not belong to $f(x)$. As a consequence, $E \neq f(x)$ for all $x \in X$. Hence the map f is not onto. In particular, it is not bijective.