MATH 409–501/503 Fall 2013

# Sample problems for Test 2

**Any problem may be altered or replaced by a different one!**

**Problem 1 (20 pts.)** Prove the Chain Rule: if a function $f$ is differentiable at a point $c$ and a function $g$ is differentiable at $f(c)$, then the composition $g \circ f$ is differentiable at $c$ and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

**Problem 2 (25 pts.)** Find the following limits of functions:

(i) $\lim\limits_{x \to 0} (1+x)^{1/x}$, (ii) $\lim\limits_{x \to +\infty} (1+x)^{1/x}$, (iii) $\lim\limits_{x \to 0+} x^x$.

**Problem 3 (20 pts.)** Find the limit of a sequence

$$x_n = \frac{1^k + 2^k + \dots + n^k}{n^{k+1}}, \quad n = 1, 2, \dots,$$

where $k$ is a natural number.

**Problem 4 (25 pts.)** Find indefinite integrals and evaluate definite integrals:

(i) $\displaystyle\int \frac{x^2}{1-x}\,dx$, (ii) $\displaystyle\int_0^\pi \sin^2(2x)\,dx$, (iii) $\displaystyle\int \log^3 x\,dx$,

(iv) $\displaystyle\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$, (v) $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$.

**Bonus Problem 5 (15 pts.)** Suppose that a function $p : \mathbb{R} \to \mathbb{R}$ is locally a polynomial, which means that for every $c \in \mathbb{R}$ there exists $\varepsilon > 0$ such that $p$ coincides with a polynomial on the interval $(c-\varepsilon, c+\varepsilon)$. Prove that $p$ is a polynomial.

**Bonus Problem 6 (15 pts.)** Show that a function

$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{1-x^2}\right) & \text{if } |x| < 1, \\ 0 \quad \text{if } |x| \geq 1 \end{cases}$$

is infinitely differentiable on $\mathbb{R}$.