Math 412-501 September 22, 2006

Exam 1

Problem 1 (40 pts.) Let $f(x) = 2x$ for $0 \le x \le \pi$.

(i) Find the Fourier sine series of f (with $[0, \pi]$ as the basic interval).

(ii) Over the interval $[-2.5\pi, 2.5\pi]$, sketch the function to which the series converges.

(iii) Roughly sketch the 12th partial sum of the series.

(iv) Briefly describe how the answers to (ii) and (iii) would change if we studied the Fourier cosine series instead.

Problem 2 (30 pts.) Solve the initial-boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \pi, \quad t > 0),$$

$$u(x,0) = 2x \qquad (0 < x < \pi),$$

$$u(0,t) = u(\pi,t) = 0 \qquad (t > 0).$$

You may stop when you can say "And now continue as in Problem 1, above."

Problem 3 (30 pts.) Consider the initial value problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\,\frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty, \quad -\infty < t < \infty),$$

$$u(x,0) = 3\sin \pi x \qquad (-\infty < x < \infty),$$

$$\frac{\partial u}{\partial t}(x,0) = 8\pi \sin 3\pi x \qquad (-\infty < x < \infty).$$

(i) Solve the problem (try to obtain a simple formula).

(ii) Determine which two of the following boundary conditions are satisfied by the solution:

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(0,t) = 0, \quad u(0.5,t) = 0, \quad \frac{\partial u}{\partial x}(0.5,t) = 0.$$

Bonus Problem 4 (35 pts.) Consider the initial-boundary value problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (x > 0, \quad t > 0),$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = 0 \qquad (x > 0),$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad (t > 0),$$

where $f(x) = 4\sin^2 \pi x$ for $2 \le x \le 3$ and $f(x) = 0$ otherwise.

(i) Sketch the solution $u(x,t)$ as a function of x for $t = 0$, $t = 1$, $t = 3$, and $t = 4$.

(ii) Describe how the answer to (i) would change if we considered the boundary condition $u(0,t) = 0$ instead.