Math 412-501 October 20, 2006

Exam 2: Solutions

Problem 1 (50 pts.) Solve the heat equation in a rectangle $0 < x < \pi$, $0 < y < \pi$,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

subject to the initial condition

$$u(x, y, 0) = (\sin 2x + \sin 3x) \sin y$$

and the boundary conditions

$$u(0, y, t) = u(\pi, y, t) = 0, \qquad u(x, 0, t) = u(x, \pi, t) = 0.$$

Solution: $u(x, y, t) = e^{-5t} \sin 2x \sin y + e^{-10t} \sin 3x \sin y$.

We search for the solution of the initial-boundary value problem as a superposition of solutions $u(x, y, t) = \phi(x)h(y)G(t)$ with separated variables of the heat equation that satisfy the boundary conditions. Substituting $u(x, y, t) = \phi(x)h(y)G(t)$ into the heat equation, we obtain

$$\phi(x)h(y)G'(t) = \phi''(x)h(y)G(t) + \phi(x)h''(y)G(t),$$

$$\frac{G'(t)}{G(t)} = \frac{\phi''(x)}{\phi(x)} + \frac{h''(y)}{h(y)}.$$

Since any of the expressions $\frac{G'(t)}{G(t)}$, $\frac{\phi''(x)}{\phi(x)}$, and $\frac{h''(y)}{h(y)}$ depend on one of the variables x, y, t and does not depend on the other two, it follows that each of these expressions is constant. Hence

$$\frac{\phi''(x)}{\phi(x)} = -\lambda, \qquad \frac{h''(y)}{h(y)} = -\mu, \qquad \frac{G'(t)}{G(t)} = -(\lambda + \mu),$$

where λ and μ are constants. Then

$$\phi'' = -\lambda\phi, \qquad h'' = -\mu h, \qquad G' = -(\lambda + \mu)G.$$

Conversely, if functions ϕ, h, and G are solutions of the above ODEs for the same values of λ and μ, then $u(x, y, t) = \phi(x)h(y)G(t)$ is a solution of the heat equation.

Substituting $u(x, y, t) = \phi(x)h(y)G(t)$ into the boundary conditions, we get

$$\phi(0)h(y)G(t) = \phi(\pi)h(y)G(t) = 0, \qquad \phi(x)h(0)G(t) = \phi(x)h(\pi)G(t) = 0.$$

It is no loss to assume that neither ϕ nor h nor G is identically zero. Then the boundary conditions are satisfied if and only if $\phi(0) = \phi(\pi) = 0$, $h(0) = h(\pi) = 0$.

To determine ϕ, we have an eigenvalue problem

$$\phi'' = -\lambda\phi, \qquad \phi(0) = \phi(\pi) = 0.$$

This problem has eigenvalues $\lambda_n = n^2$, $n = 1, 2, \dots$. The corresponding eigenfunctions are $\phi_n(x) = \sin nx$.

To determine h, we have the same eigenvalue problem

$$h'' = -\mu h, \qquad h(0) = h(\pi) = 0.$$

Hence the eigenvalues are $\mu_m = m^2$, $m = 1, 2, \ldots$. The corresponding eigenfunctions are $h_m(y) = \sin my$.

The function G is to be determined from the equation $G' = -(\lambda + \mu)G$. The general solution of this equation is $G(t) = c_0 e^{-(\lambda+\mu)t}$, where c_0 is a constant.

Thus we obtain the following solutions of the heat equation satisfying the boundary conditions:

$$u_{nm}(x, y, t) = e^{-(\lambda_n+\mu_m)t}\phi_n(x)h_m(y) = e^{-(n^2+m^2)t} \sin nx \sin my, \qquad n, m = 1, 2, 3, \ldots$$

A superposition of these solutions is a double series

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm} e^{-(n^2+m^2)t} \sin nx \sin my,$$

where c_{nm} are constants. To determine the coefficients c_{nm}, we substitute the series into the initial condition $u(x, y, 0) = (\sin 2x + \sin 3x) \sin y$:

$$(\sin 2x + \sin 3x) \sin y = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm} \sin nx \sin my.$$

It is easy to observe that $c_{2,1} = c_{3,1} = 1$ while the other coefficients are equal to 0. Therefore

$$u(x, y, t) = e^{-5t} \sin 2x \sin y + e^{-10t} \sin 3x \sin y.$$

Problem 2 (50 pts.) Solve Laplace's equation inside a quarter-circle $0 < r < 1$, $0 < \theta < \pi/2$ (in polar coordinates r, θ) subject to the boundary conditions

$$u(r, 0) = 0, \qquad u(r, \pi/2) = 0, \qquad |u(0, \theta)| < \infty, \qquad u(1, \theta) = f(\theta).$$

Solution: $u(r, \theta) = \sum_{n=1}^{\infty} c_n r^{2n} \sin 2n\theta$, where

$$c_n = \frac{4}{\pi}\int_0^{\pi/2} f(\theta) \sin 2n\theta \, d\theta, \quad n = 1, 2, \ldots$$

Laplace's equation in polar coordinates (r, θ):

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

We search for the solution of the boundary value problem as a superposition of solutions $u(r, \theta) = h(r)\phi(\theta)$ with separated variables of Laplace's equation that satisfy the three homogeneous boundary conditions. Substituting $u(r, \theta) = h(r)\phi(\theta)$ into Laplace's equation, we obtain

$$h''(r)\phi(\theta) + \frac{1}{r}\, h'(r)\phi(\theta) + \frac{1}{r^2}\, h(r)\phi''(\theta) = 0,$$

$$\frac{r^2 h''(r) + r h'(r)}{h(r)} = -\frac{\phi''(\theta)}{\phi(\theta)}.$$

Since the left-hand side does not depend on θ while the right-hand side does not depend on r, it follows that

$$\frac{r^2h''(r) + rh'(r)}{h(r)} = -\frac{\phi''(\theta)}{\phi(\theta)} = \lambda,$$

where λ is a constant. Then

$$r^2h''(r) + rh'(r) = \lambda h(r), \qquad \phi'' = -\lambda\phi.$$

Conversely, if functions h and ϕ are solutions of the above ODEs for the same value of λ, then $u(r,\theta) = h(r)\phi(\theta)$ is a solution of Laplace's equation in polar coordinates.

Substituting $u(r,\theta) = h(r)\phi(\theta)$ into the homogeneous boundary conditions, we get

$$h(r)\phi(0) = 0, \quad h(r)\phi(\pi/2) = 0, \quad |h(0)\phi(\theta)| < \infty.$$

It is no loss to assume that neither h nor ϕ is identically zero. Then the boundary conditions are satisfied if and only if $\phi(0) = \phi(\pi/2) = 0$, $|h(0)| < \infty$.

To determine ϕ, we have an eigenvalue problem

$$\phi'' = -\lambda\phi, \qquad \phi(0) = \phi(\pi/2) = 0.$$

This problem has eigenvalues $\lambda_n = (2n)^2$, $n = 1, 2, \dots$. The corresponding eigenfunctions are $\phi_n(\theta) = \sin 2n\theta$.

The function h is to be determined from the equation $r^2h'' + rh' = \lambda h$ and the boundary condition $|h(0)| < \infty$. We may assume that λ is one of the above eigenvalues so that $\lambda > 0$. Then the general solution of the equation is $h(r) = c_1r^\mu + c_2r^{-\mu}$, where $\mu = \sqrt{\lambda}$ and c_1, c_2 are constants. The boundary condition $|h(0)| < \infty$ holds if $c_2 = 0$.

Thus we obtain the following solutions of Laplace's equation satisfying the three homogeneous boundary conditions:

$$u_n(r,\theta) = r^{2n}\sin 2n\theta, \qquad n = 1, 2, \dots$$

A superposition of these solutions is a series

$$u(r,\theta) = \sum_{n=1}^{\infty} c_n r^{2n} \sin 2n\theta,$$

where $c_1, c_2, \dots$ are constants. Substituting the series into the boundary condition $u(1,\theta) = f(\theta)$, we get

$$f(\theta) = \sum_{n=1}^{\infty} c_n \sin 2n\theta.$$

The right-hand side is a Fourier sine series on the interval $[0, \pi/2]$. Therefore the boundary condition is satisfied if this is the Fourier sine series of the function $f(\theta)$ on $[0, \pi/2]$. Hence

$$c_n = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\sin 2n\theta\, d\theta, \quad n = 1, 2, \dots$$

Bonus Problem 3 (40 pts.) Consider a regular Sturm-Liouville eigenvalue problem

$$\phi'' + \lambda\phi = 0, \qquad \phi'(0) = 0, \quad \phi'(1) + h\phi(1) = 0,$$

where h is a real constant.

(i) For what values of h is $\lambda = 0$ an eigenvalue?

Solution: $h = 0$.

In the case $\lambda = 0$, the general solution of the equation $\phi'' + \lambda\phi = 0$ is a linear function $\phi(x) = c_1 x + c_2$, where c_1, c_2 are constants. Substituting it into the boundary conditions $\phi'(0) = 0$ and $\phi'(1) + h\phi(1) = 0$, we obtain equalities $c_1 = 0$, $c_1 + h(c_1 + c_2) = 0$. They imply that $c_1 = hc_2 = 0$. If $h \neq 0$, it follows that $c_1 = c_2 = 0$, hence there are no eigenfunctions with eigenvalue $\lambda = 0$. If $h = 0$ then $\phi(x) = 1$ is indeed an eigenfunction.

(ii) For what values of h are all eigenvalues positive?

Solution: $h > 0$.

In the case $\lambda < 0$, the general solution of the equation $\phi'' + \lambda\phi = 0$ is

$$\phi(x) = c_1 \cosh \mu x + c_2 \sinh \mu x,$$

where $\mu = \sqrt{-\lambda} > 0$ and c_1, c_2 are constants. Note that

$$\phi'(x) = c_1 \mu \sinh \mu x + c_2 \mu \cosh \mu x.$$

The boundary condition $\phi'(0) = 0$ is satisfied if and only if $c_2 = 0$. Substituting $\phi(x) = c_1 \cosh \mu x$ into the boundary condition $\phi'(1) + h\phi(1) = 0$, we obtain

$$c_1 \mu \sinh \mu + h c_1 \cosh \mu = 0,$$

$$c_1(\mu \tanh \mu + h) = 0.$$

If $\mu \tanh \mu \neq -h$, it follows that $c_1 = 0$, hence there are no eigenfunctions with eigenvalue $\lambda = -\mu^2$. If $\mu \tanh \mu = -h$ then $\phi(x) = \cosh \mu x$ is indeed an eigenfunction.

The function $f(\mu) = \mu \tanh \mu$ is continuous. It is easy to see that $f(0) = 0$ and $f(\mu) > 0$ for $\mu > 0$. Since $\tanh \mu \to 1$ as $\mu \to +\infty$, we have that $f(\mu) \to +\infty$ as $\mu \to +\infty$. It follows that f takes all positive values on $(0, \infty)$.

By the above the eigenvalue problem has a negative eigenvalue if and only if $h < 0$. As shown in the solution to the part (i), $\lambda = 0$ is an eigenvalue only for $h = 0$. Hence all eigenvalues are positive if and only if $h > 0$.

The fact that for any $h \geq 0$ all eigenvalues are nonnegative can also be obtained using the Rayleigh quotient. If ϕ is an eigenfunction corresponding to an eigenvalue λ then

$$\lambda = \frac{-\phi\phi' \big|_0^1 + \displaystyle\int_0^1 |\phi'(x)|^2\, dx}{\displaystyle\int_0^1 |\phi(x)|^2\, dx}.$$

The boundary conditions imply that

$$-\phi\phi' \big|_0^1 = \phi(0)\phi'(0) - \phi(1)\phi'(1) = h|\phi(1)|^2.$$

Hence $\lambda \geq 0$ provided that $h \geq 0$.

(iii) How many negative eigenvalues can this problem have?

Solution: One negative eigenvalue for $h < 0$.

Let $f(\mu) = \mu \tanh \mu$. As shown in the solution to the part (ii), $\lambda < 0$ is an eigenvalue if and only if $f(\mu) = -h$, where $\lambda = -\mu^2$, $\mu > 0$. Observe that

$$f'(\mu) = \tanh \mu + \mu \tanh' \mu = \tanh \mu + \frac{\mu}{\cosh^2 \mu}.$$

In particular, $f'(\mu) > 0$ for $\mu > 0$. Since f is continuous, $f(0) = 0$, and $f(\mu) \to +\infty$ as $\mu \to +\infty$, it follows that f is a one-to-one map of the interval $(0, \infty)$ onto itself. Therefore for any $h < 0$ the eigenvalue problem has exactly one negative eigenvalue.

(iv) Find an equation for positive eigenvalues.

Solution: $\tan\sqrt{\lambda} = \dfrac{h}{\sqrt{\lambda}}$.

In the case $\lambda > 0$, the general solution of the equation $\phi'' + \lambda\phi = 0$ is

$$\phi(x) = c_1 \cos\mu x + c_2 \sin\mu x,$$

where $\mu = \sqrt{\lambda}$ and c_1, c_2 are constants. Note that

$$\phi'(x) = -c_1\mu\sin\mu x + c_2\mu\cos\mu x.$$

The boundary condition $\phi'(0) = 0$ is satisfied if and only if $c_2 = 0$. Substituting $\phi(x) = c_1\cos\mu x$ into the boundary condition $\phi'(1) + h\phi(1) = 0$, we obtain

$$-c_1\mu\sin\mu + hc_1\cos\mu = 0,$$

$$c_1(h\cos\mu - \mu\sin\mu) = 0.$$

If $h\cos\mu \neq \mu\sin\mu$, it follows that $c_1 = 0$, hence there are no eigenfunctions with eigenvalue $\lambda = \mu^2$. If $h\cos\mu = \mu\sin\mu$ then $\phi(x) = \cos\mu x$ is indeed an eigenfunction.

Thus $h\cos\sqrt{\lambda} = \sqrt{\lambda}\sin\sqrt{\lambda}$ is an equation for positive eigenvalues. Note that for any positive solution λ of this equation we have $\cos\sqrt{\lambda} \neq 0$. Indeed, if $\cos\sqrt{\lambda} = 0$ then $\sin\sqrt{\lambda} = \pm 1$ and $\sqrt{\lambda}\sin\sqrt{\lambda} \neq 0$. It follows that for $\lambda > 0$ this equation is equivalent to

$$\tan\sqrt{\lambda} = \frac{h}{\sqrt{\lambda}}.$$

(v) Find the asymptotics of λ_n as $n \to \infty$.

Solution: $\sqrt{\lambda_n} \approx (n-1)\pi$ as $n \to \infty$.

Positive eigenvalues are found from the equation $\tan\sqrt{\lambda} = h/\sqrt{\lambda}$. The function $f_1(\mu) = \tan\mu$ is continuous, strictly increasing and assumes all real values on each of the intervals $(\pi m - \pi/2, \pi m + \pi/2)$, $m = 0, 1, 2, \dots$.

In the case $h > 0$, the function $f_2(\mu) = h/\mu$ is continuous and strictly decreasing on $(0, \infty)$. It follows that the equation $f_1(\mu) = f_2(\mu)$ has exactly one solution in each of the intervals $(0, \pi/2)$ and $(\pi m - \pi/2, \pi m + \pi/2)$, $m = 1, 2, \dots$. In this case all eigenvalues are positive, hence $\pi(n-1) - \pi/2 < \sqrt{\lambda_n} < \pi(n-1) + \pi/2$. Moreover, since $\tan\sqrt{\lambda_n} \to 0$ as $n \to \infty$, it follows that $\sqrt{\lambda_n} \approx (n-1)\pi$.

If $h = 0$ then $\lambda_n = \big((n-1)\pi\big)^2$, $n = 1, 2, \dots$.

If $h < 0$ then $\lambda_1 < 0 < \lambda_2$. In this case the function $f_2(\mu) = h/\mu$ is negative and strictly increasing on $(0, \infty)$. The equation $f_1(\mu) = f_2(\mu)$ has no solution in $(0, \pi/2)$ and exactly one solution in each of the intervals $(\pi m - \pi/2, \pi m + \pi/2)$, $m = 1, 2, \dots$. We conclude that $\pi(n-1) - \pi/2 < \sqrt{\lambda_n} < \pi(n-1) + \pi/2$ for $n \geq 2$. It follows that $\sqrt{\lambda_n} \approx (n-1)\pi$ in this case as well.