Math 412-501 Fall 2006

Sample problems for the final exam: Solutions

Any problem may be altered or replaced by a different one!

Some possibly useful information

- Parseval's equality for the complex form of the Fourier series on $(-\pi, \pi)$:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad \Longrightarrow \qquad \int_{-\pi}^{\pi} |f(x)|^2\, dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2.$$

- Fourier sine and cosine transforms of the second derivative:

$$S[f''](\omega) = \frac{2}{\pi} f(0)\,\omega - \omega^2 S[f](\omega), \qquad C[f''](\omega) = -\frac{2}{\pi} f'(0) - \omega^2 C[f](\omega).$$

- Laplace's operator in polar coordinates r, θ:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

- Any nonzero solution of a regular Sturm-Liouville equation

$$(p\phi')' + q\phi + \lambda\sigma\phi = 0 \qquad (a < x < b)$$

satisfies the Rayleigh quotient relation

$$\lambda = \frac{-p\phi\phi'\Big|_a^b + \int_a^b \big(p(\phi')^2 - q\phi^2\big)\, dx}{\int_a^b \phi^2\sigma\, dx}.$$

- Some table integrals:

$$\int x^2 e^{iax}\, dx = \left(\frac{x^2}{ia} + \frac{2x}{a^2} - \frac{2}{ia^3}\right) e^{iax} + C, \quad a \neq 0;$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} e^{i\beta x}\, dx = \sqrt{\frac{\pi}{\alpha}}\, e^{-\beta^2/(4\alpha)}, \quad \alpha > 0,\ \beta \in \mathbb{R};$$

$$\int_{-\infty}^{\infty} e^{-\alpha|x|} e^{i\beta x}\, dx = \frac{2\alpha}{\alpha^2 + \beta^2}, \quad \alpha > 0,\ \beta \in \mathbb{R}.$$

Problem 1 Let $f(x) = x^2$.

(i) Find the Fourier series (complex form) of $f(x)$ on the interval $(-\pi, \pi)$.

The required series is $\displaystyle\sum_{n=-\infty}^{\infty} c_n e^{inx}$, where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

In particular,

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\frac{x^3}{3}\Big|_{x=-\pi}^{\pi} = \frac{1}{2\pi}\frac{2\pi^3}{3} = \frac{\pi^2}{3}.$$

If $n \neq 0$ then we have to integrate by parts twice:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{x^2(e^{-inx})'}{-in}\,dx = \frac{1}{2\pi}\frac{x^2 e^{-inx}}{-in}\Big|_{-\pi}^{\pi} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2xe^{-inx}}{in}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2xe^{-inx}}{in}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2x(e^{-inx})'}{-(in)^2}\,dx = \frac{1}{2\pi}\frac{2xe^{-inx}}{-(in)^2}\Big|_{-\pi}^{\pi} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2e^{-inx}}{(in)^2}\,dx$$

$$= \frac{e^{-in\pi} + e^{in\pi}}{n^2} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2e^{-inx}}{(in)^2}\,dx = \frac{2(-1)^n}{n^2} + \frac{1}{2\pi}\frac{2e^{-inx}}{-(in)^3}\Big|_{-\pi}^{\pi} = \frac{2(-1)^n}{n^2}.$$

To save time, we could instead use the table integral

$$\int x^2 e^{iax}\,dx = \left(\frac{x^2}{ia} + \frac{2x}{a^2} - \frac{2}{ia^3}\right)e^{iax} + C, \quad a \neq 0.$$

According to this integral,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx = \frac{1}{2\pi}\left(-\frac{x^2}{in} + \frac{2x}{n^2} + \frac{2}{in^3}\right)e^{-inx}\Big|_{-\pi}^{\pi} = \frac{1}{2\pi}\frac{2\pi(e^{-in\pi} + e^{in\pi})}{n^2} = \frac{2(-1)^n}{n^2}.$$

Thus

$$x^2 \sim \frac{\pi^2}{3} + \sum_{\substack{-\infty<n<\infty \\ n\neq 0}} \frac{2(-1)^n}{n^2}e^{inx}.$$

(ii) Rewrite the Fourier series of $f(x)$ in the real form.

$$\frac{\pi^2}{3} + \sum_{\substack{-\infty<n<\infty \\ n\neq 0}} \frac{2(-1)^n}{n^2}e^{inx} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}(e^{inx} + e^{-inx}) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx.$$

Thus

$$x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx.$$

(iii) Sketch the function to which the Fourier series converges.

The series converges to the 2π-periodic function that coincides with $f(x)$ for $-\pi \le x \le \pi$. The sum is continuous and piecewise smooth hence the convergence is uniform. The derivative of the sum has jump discontinuities at points $\pi + 2k\pi$, $k \in \mathbb{Z}$. The graph is a scalloped curve.

(iv) Use Parseval's equality to evaluate $\sum_{n=1}^{\infty} n^{-4}$.

In our case, Parseval's equality can be written as

$$\langle f, f\rangle = \sum_{n=-\infty}^{\infty} \frac{|\langle f, \phi_n\rangle|^2}{\langle \phi_n, \phi_n\rangle},$$

where

$$\langle g, h\rangle = \int_{-\pi}^{\pi} g(x)\overline{h(x)}\, dx$$

and $\phi_n(x) = e^{inx}$. Since $c_n = \frac{\langle f,\phi_n\rangle}{\langle \phi_n,\phi_n\rangle}$ and $\langle \phi_n, \phi_n\rangle = 2\pi$ for all $n \in \mathbb{Z}$, it can be reduced to an equivalent form

$$\int_{-\pi}^{\pi} |f(x)|^2\, dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Now

$$\int_{-\pi}^{\pi} |f(x)|^2\, dx = \int_{-\pi}^{\pi} x^4\, dx = \frac{x^5}{5}\Big|_{-\pi}^{\pi} = \frac{2\pi^5}{5},$$

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty} \frac{4}{n^4}.$$

Therefore

$$\frac{1}{2\pi}\,\frac{2\pi^5}{5} = \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty} \frac{4}{n^4}.$$

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{8}\left(\frac{\pi^4}{5} - \frac{\pi^4}{9}\right) = \frac{\pi^4}{90}.$$

Problem 2 Solve Laplace's equation in a disk,

$$\nabla^2 u = 0 \quad (0 \le r < a), \qquad u(a, \theta) = f(\theta).$$

Laplace's operator in polar coordinates r, θ:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

We search for the solution of the boundary value problem as a superposition of solutions $u(r, \theta) = h(r)\phi(\theta)$ $(0 < r < a, -\pi < \theta < \pi)$ with separated variables of Laplace's equation in the disk. Solutions with separated variables satisfy periodic boundary conditions

$$u(r, -\pi) = u(r, \pi), \qquad \frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi)$$

and the singular boundary condition

$$|u(0, \theta)| < \infty.$$

Substituting $u(r,\theta) = h(r)\phi(\theta)$ into Laplace's equation, we obtain

$$h''(r)\phi(\theta) + \frac{1}{r}\,h'(r)\phi(\theta) + \frac{1}{r^2}\,h(r)\phi''(\theta) = 0,$$

$$\frac{r^2h''(r) + rh'(r)}{h(r)} = -\frac{\phi''(\theta)}{\phi(\theta)}.$$

Since the left-hand side does not depend on θ while the right-hand side does not depend on r, it follows that

$$\frac{r^2h''(r) + rh'(r)}{h(r)} = -\frac{\phi''(\theta)}{\phi(\theta)} = \lambda,$$

where λ is a constant. Then

$$r^2h''(r) + rh'(r) = \lambda h(r), \qquad \phi'' = -\lambda\phi.$$

Conversely, if functions h and ϕ are solutions of the above ODEs for the same value of λ, then $u(r,\theta) = h(r)\phi(\theta)$ is a solution of Laplace's equation in polar coordinates.

Substituting $u(r,\theta) = h(r)\phi(\theta)$ into the periodic and singular boundary conditions, we get

$$h(r)\phi(-\pi) = h(r)\phi(\pi), \quad h(r)\phi'(-\pi) = h(r)\phi'(\pi), \quad |h(0)\phi(\theta)| < \infty.$$

It is no loss to assume that neither h nor ϕ is identically zero. Then the boundary conditions are satisfied if and only if $\phi(-\pi) = \phi(\pi)$, $\phi'(-\pi) = \phi'(\pi)$, $|h(0)| < \infty$.

To determine ϕ, we have an eigenvalue problem

$$\phi'' = -\lambda\phi, \qquad \phi(-\pi) = \phi(\pi), \quad \phi'(-\pi) = \phi'(\pi).$$

This problem has eigenvalues $\lambda_n = n^2$, $n = 0, 1, 2, \dots$. The eigenvalue $\lambda_0 = 0$ is simple, the others are of multiplicity 2. The corresponding eigenfunctions are $\phi_0 = 1$, $\phi_n(\theta) = \cos n\theta$ and $\psi_n(\theta) = \sin n\theta$ for $n \geq 1$.

The function h is to be determined from the equation $r^2h'' + rh' = \lambda h$ and the boundary condition $|h(0)| < \infty$. We may assume that λ is one of the above eigenvalues so that $\lambda \geq 0$. If $\lambda > 0$ then the general solution of the equation is $h(r) = c_1 r^{\mu} + c_2 r^{-\mu}$, where $\mu = \sqrt{\lambda}$ and c_1, c_2 are constants. In the case $\lambda = 0$, the general solution is $h(r) = c_1 + c_2 \log r$, where c_1, c_2 are constants. In either case, the boundary condition $|h(0)| < \infty$ holds if $c_2 = 0$.

Thus we obtain the following solutions of Laplace's equation in the disk:

$$u_0 = 1, \qquad u_n(r,\theta) = r^n \cos n\theta, \quad \tilde{u}_n(r,\theta) = r^n \sin n\theta, \quad n = 1, 2, \dots$$

A superposition of these solutions is a series

$$u(r,\theta) = \alpha_0 + \sum\nolimits_{n=1}^{\infty} r^n(\alpha_n \cos n\theta + \beta_n \sin n\theta),$$

where $\alpha_0, \alpha_1, \dots$ and $\beta_1, \beta_2, \dots$ are constants. Substituting the series into the boundary condition $u(a,\theta) = f(\theta)$, we get

$$f(\theta) = \alpha_0 + \sum\nolimits_{n=1}^{\infty} a^n(\alpha_n \cos n\theta + \beta_n \sin n\theta).$$

The right-hand side is a Fourier series on the interval $(-\pi,\pi)$. Therefore the boundary condition is satisfied if the right-hand side coincides with the Fourier series

$$A_0 + \sum\nolimits_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)$$

of the function $f(\theta)$ on $(-\pi,\pi)$. Hence

$$\alpha_0 = A_0, \qquad \alpha_n = a^{-n}A_n, \quad \beta_n = a^{-n}B_n, \quad n = 1, 2, \dots$$

and

$$u(r,\theta) = A_0 + \sum\nolimits_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (A_n \cos n\theta + B_n \sin n\theta),$$

where

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta, \qquad A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta, \quad B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta, \quad n = 1,2,\dots$$

Problem 3 Find Green's function for the boundary value problem

$$\frac{d^2u}{dx^2} - u = f(x) \quad (0 < x < 1), \qquad u'(0) = u'(1) = 0.$$

The Green function $G(x, x_0)$ should satisfy

$$\frac{\partial^2 G}{\partial x^2} - G = \delta(x - x_0), \qquad \frac{\partial G}{\partial x}(0, x_0) = \frac{\partial G}{\partial x}(1, x_0) = 0.$$

Since $\dfrac{\partial^2 G}{\partial x^2} - G = 0$ for $x < x_0$ and $x > x_0$, it follows that

$$G(x, x_0) = \begin{cases} ae^x + be^{-x} & \text{for } x < x_0, \\ ce^x + de^{-x} & \text{for } x > x_0, \end{cases}$$

where constants a, b, c, d may depend on x_0. Then

$$\frac{\partial G}{\partial x}(x, x_0) = \begin{cases} ae^x - be^{-x} & \text{for } x < x_0, \\ ce^x - de^{-x} & \text{for } x > x_0. \end{cases}$$

The boundary conditions imply that $a = b$ and $ce = de^{-1}$.

Now impose the gluing conditions at $x = x_0$, that is, continuity of the function and jump discontinuity of the first derivative:

$$G(x, x_0)\Big|_{x=x_0-} = G(x, x_0)\Big|_{x=x_0+}, \qquad \frac{\partial G}{\partial x}\Big|_{x=x_0+} - \frac{\partial G}{\partial x}\Big|_{x=x_0-} = 1.$$

The two conditions imply that

$$ae^{x_0} + be^{-x_0} = ce^{x_0} + de^{-x_0}, \qquad ce^{x_0} - de^{-x_0} - (ae^{x_0} - be^{-x_0}) = 1.$$

Since $b = a$ and $d = ce^2$, we get

$$a(e^{x_0} + e^{-x_0}) = c(e^{x_0} + e^{2-x_0}), \qquad c(e^{x_0} - e^{2-x_0}) - a(e^{x_0} - e^{-x_0}) = 1.$$

Then

$$\begin{aligned} e^{x_0} + e^{-x_0} &= c(e^{x_0} - e^{2-x_0})(e^{x_0} + e^{-x_0}) - a(e^{x_0} - e^{-x_0})(e^{x_0} + e^{-x_0}) \\ &= c(e^{x_0} - e^{2-x_0})(e^{x_0} + e^{-x_0}) - c(e^{x_0} + e^{2-x_0})(e^{x_0} - e^{-x_0}) = 2c(1 - e^2). \end{aligned}$$

Therefore

$$c = \frac{e^{x_0} + e^{-x_0}}{2(1 - e^2)}, \qquad a = c\,\frac{e^{x_0} + e^{2-x_0}}{e^{x_0} + e^{-x_0}} = \frac{e^{x_0} + e^{2-x_0}}{2(1 - e^2)},$$

$$d = ce^2 = \frac{e^{x_0} + e^{-x_0}}{2(e^{-2} - 1)}, \qquad b = a = \frac{e^{x_0} + e^{2-x_0}}{2(1 - e^2)}.$$

Finally,

$$G(x,x_0) = \begin{cases} \dfrac{(e^{x_0}+e^{2-x_0})(e^x+e^{-x})}{2(1-e^2)} & \text{for } \ x<x_0, \\ \dfrac{(e^{x_0}+e^{-x_0})(e^x+e^{2-x})}{2(1-e^2)} & \text{for } \ x>x_0. \end{cases}$$

Observe that $G(x,x_0) = G(x_0,x)$.

Problem 4 Solve the initial-boundary value problem for the heat equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0<x<\pi, \quad t>0),$$

$$u(x,0) = f(x) \qquad (0<x<\pi),$$

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(\pi,t) + 2u(\pi,t) = 0.$$

In the process you will discover a sequence of eigenfunctions and eigenvalues, which you should name $\phi_n(x)$ and λ_n. Describe the λ_n qualitatively (e.g., find an equation for them) but do not expect to find their exact numerical values. Also, do not bother to evaluate normalization integrals for ϕ_n.

We search for the solution of the initial-boundary value problem as a superposition of solutions $u(x,t) = \phi(x)g(t)$ with separated variables of the heat equation that satisfy the boundary conditions.

Substituting $u(x,t) = \phi(x)g(t)$ into the heat equation, we obtain

$$\phi(x)g'(t) = \phi''(x)g(t),$$

$$\frac{g'(t)}{g(t)} = \frac{\phi''(x)}{\phi(x)}.$$

Since the left-hand side does not depend on x while the right-hand side does not depend on t, it follows that

$$\frac{g'(t)}{g(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda,$$

where λ is a constant. Then

$$g' = -\lambda g, \qquad \phi'' = -\lambda\phi.$$

Conversely, if functions g and ϕ are solutions of the above ODEs for the same value of λ, then $u(x,t) = \phi(x)g(t)$ is a solution of the heat equation.

Substituting $u(x,t) = \phi(x)g(t)$ into the boundary conditions, we get

$$\phi(0)g(t) = 0, \qquad \phi'(\pi)g(t) + 2\phi(\pi)g(t) = 0.$$

It is no loss to assume that g is not identically zero. Then the boundary conditions are satisfied if and only if $\phi(0) = 0$, $\phi'(\pi) + 2\phi(\pi) = 0$.

To determine ϕ, we have an eigenvalue problem

$$\phi'' = -\lambda\phi, \qquad \phi(0) = 0, \quad \phi'(\pi) + 2\phi(\pi) = 0.$$

This is a regular Sturm-Liouville eigenvalue problem. If ϕ is an eigenfunction corresponding to an eigenvalue λ, then the Rayleigh quotient relation holds:

$$\lambda = \frac{-\phi\phi'\Big|_0^\pi + \displaystyle\int_0^\pi |\phi'(x)|^2\,dx}{\displaystyle\int_0^\pi |\phi(x)|^2\,dx}.$$

Note that $-\phi\phi'\,\big|_0^\pi = \phi(0)\phi'(0) - \phi(\pi)\phi'(\pi) = 2|\phi(\pi)|^2$. It follows that $\lambda \geq 0$. Moreover, $\lambda > 0$ since constants are not eigenfunctions. Hence all eigenvalues are positive.

For any $\lambda > 0$ the general solution of the equation $\phi'' = -\lambda\phi$ is

$$\phi(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x),$$

where c_1, c_2 are constants. The boundary condition $\phi(0) = 0$ holds if $c_1 = 0$. Then the condition $\phi'(\pi) + 2\phi(\pi) = 0$ holds if

$$c_2\Big(\sqrt{\lambda}\cos(\sqrt{\lambda}\,\pi) + 2\sin(\sqrt{\lambda}\,\pi)\Big) = 0.$$

A nonzero solution exists if

$$\sqrt{\lambda}\,\cos(\sqrt{\lambda}\,\pi) + 2\sin(\sqrt{\lambda}\,\pi) = 0 \quad\Longrightarrow\quad -\frac{1}{2}\sqrt{\lambda} = \tan(\sqrt{\lambda}\,\pi).$$

It follows that the eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$ are solutions of the equation $-\frac{1}{2}\sqrt{\lambda} = \tan(\sqrt{\lambda}\,\pi)$, and the corresponding eigenfunctions are $\phi_n(x) = \sin(\sqrt{\lambda_n}\, x)$.

The function g is to be determined from the equation $g' = -\lambda g$. The general solution is $g(t) = c_0 e^{-\lambda t}$, where c_0 is a constant.

Thus we obtain the following solutions of the heat equation that satisfy the boundary conditions:

$$u_n(x,t) = e^{-\lambda_n t}\phi_n(x) = e^{-\lambda_n t}\sin(\sqrt{\lambda_n}\, x), \quad n = 1, 2, \dots$$

A superposition of these solutions is a series

$$u(x,t) = \sum\nolimits_{n=1}^{\infty} c_n e^{-\lambda_n t}\phi_n(x) = \sum\nolimits_{n=1}^{\infty} c_n e^{-\lambda_n t}\sin(\sqrt{\lambda_n}\, x),$$

where $c_1, c_2, \dots$ are constants. Substituting the series into the initial condition $u(x,0) = f(x)$, we get

$$f(x) = \sum\nolimits_{n=1}^{\infty} c_n\phi_n(x).$$

The right-hand side is a generalized Fourier series. Therefore the initial condition is satisfied if the right-hand side coincides with the generalized Fourier series of the function f, that is, if

$$c_n = \frac{\displaystyle\int_0^\pi f(x_0)\phi_n(x_0)\,dx_0}{\displaystyle\int_0^\pi |\phi_n(x_0)|^2\,dx_0}, \qquad n = 1, 2, \dots$$

Problem 5 By the method of your choice, solve the wave equation on the half-line

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \infty,\ -\infty < t < \infty)$$

subject to

$$u(0,t) = 0, \qquad u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

Fourier's method: In view of the boundary condition, let us apply the Fourier sine transform with respect to x to both sides of the equation:

$$S\left[\frac{\partial^2 u}{\partial t^2}\right] = S\left[\frac{\partial^2 u}{\partial x^2}\right].$$

Let $U(\omega,t)$ denote the Fourier sine transform of the solution $u(x,t)$:

$$U(\omega,t) = S[u(\cdot,t)](\omega) = \frac{2}{\pi}\int_0^\infty u(x,t)\sin\omega x\,dx.$$

Then

$$S\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{\partial^2 U}{\partial t^2}, \qquad S\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{2}{\pi}\,u(0,t)\omega - \omega^2 U(\omega,t) = -\omega^2 U(\omega,t).$$

Hence

$$\frac{\partial^2 U}{\partial t^2} = -\omega^2 U(\omega,t).$$

If $\omega \neq 0$ then the general solution of the latter equation is $U(\omega,t) = a\cos\omega t + b\sin\omega t$, where $a = a(\omega)$, $b = b(\omega)$. Applying the Fourier sine transform to the initial conditions, we obtain

$$U(\omega,0) = F(\omega), \qquad \frac{\partial U}{\partial t}(\omega,0) = G(\omega),$$

where $F = S[f]$, $G = S[g]$. It follows that $a(\omega) = F(\omega)$, $b(\omega) = G(\omega)/\omega$.

Now it remains to apply the inverse Fourier sine transform:

$$u(x,t) = S^{-1}[U(\cdot,t)](x) = \int_0^\infty \left(F(\omega)\cos\omega t + \frac{G(\omega)}{\omega}\sin\omega t\right)\sin\omega x\,d\omega,$$

where

$$F(\omega) = \frac{2}{\pi}\int_0^\infty f(x_0)\sin\omega x_0\,dx_0, \qquad G(\omega) = \frac{2}{\pi}\int_0^\infty g(x_0)\sin\omega x_0\,dx_0.$$

D'Alembert's method: Define $f(x)$ and $g(x)$ for negative x to be the odd extensions of the functions given for positive x, i.e., $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all $x > 0$. By d'Alembert's formula, the function

$$u(x,t) = \frac{1}{2}\big(f(x+t) + f(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(x_0)\,dx_0$$

is the solution of the wave equation that satisfies the initial conditions

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

on the entire line. Since f and g are odd functions, it follows that $u(x,t)$ is also odd as a function of x. As a consequence, $u(0,t) = 0$ for all t. Thus the boundary condition holds as well.

Bonus Problem 6 Solve Problem 5 by a distinctly different method.

See above.

Bonus Problem 7 Find a Green function implementing the solution of Problem 2.

The solution of Problem 2:

$$u(r,\theta) = A_0 + \sum_{n=1}^\infty \left(\frac{r}{a}\right)^n (A_n\cos n\theta + B_n\sin n\theta),$$

where

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta_0)\,d\theta_0, \quad A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta_0)\cos n\theta_0\,d\theta_0, \quad B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta_0)\sin n\theta_0\,d\theta_0, \quad n = 1, 2, \ldots$$

It can be rewritten as

$$u(r,\theta) = \int_{-\pi}^{\pi} G(r,\theta;\theta_0)\,f(\theta_0)\,d\theta_0,$$

where

$$G(r,\theta;\theta_0) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n (\cos n\theta\cos n\theta_0 + \sin n\theta\sin n\theta_0)$$

is the desired Green function. The expression can be simplified:

$$\begin{aligned}
G(r,\theta;\theta_0) &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\theta_0) \\
&= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cdot \frac{e^{in(\theta-\theta_0)} + e^{-in(\theta-\theta_0)}}{2} \\
&= \frac{1}{2\pi}\sum_{n=0}^{\infty}\left(ra^{-1}e^{i(\theta-\theta_0)}\right)^n + \frac{1}{2\pi}\sum_{n=1}^{\infty}\left(ra^{-1}e^{-i(\theta-\theta_0)}\right)^n \\
&= \frac{1}{2\pi}\left(\frac{1}{1-ra^{-1}e^{i(\theta-\theta_0)}} + \frac{ra^{-1}e^{-i(\theta-\theta_0)}}{1-ra^{-1}e^{-i(\theta-\theta_0)}}\right) \\
&= \frac{1}{2\pi}\left(\frac{a}{a-re^{i(\theta-\theta_0)}} + \frac{re^{-i(\theta-\theta_0)}}{a-re^{-i(\theta-\theta_0)}}\right) \\
&= \frac{1}{2\pi}\,\frac{a^2-r^2}{(a-re^{i(\theta-\theta_0)})(a-re^{-i(\theta-\theta_0)})} \\
&= \frac{1}{2\pi}\,\frac{a^2-r^2}{a^2-2ar\cos(\theta-\theta_0)+r^2}.
\end{aligned}$$