Solutions for homework assignment #2

Problem 1. Show that the equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} + Q(u,x,t)$$

is linear if $Q(u,x,t) = \alpha(x,t)u + \beta(x,t)$ and in addition homogeneous if $\beta(x,t) = 0$.

Solution: The equation has the form $\mathcal{L}(u) = \beta(x,t)$, where $\mathcal{L}(u) = \frac{\partial u}{\partial t} - k\,\frac{\partial^2 u}{\partial x^2} - \alpha(x,t)u$. For any functions u_1, u_2 and any $r_1, r_2 \in \mathbb{R}$ we have $\mathcal{L}(r_1u_1 + r_2u_2) = r_1\mathcal{L}(u_1) + r_2\mathcal{L}(u_2)$. Hence $\mathcal{L}$ is a linear operator. Then $\mathcal{L}(u) = \beta(x,t)$ is a linear equation. If $\beta(x,t) = 0$ then the equation is linear homogeneous.

Problem 2. Show that a linear equation is homogeneous if and only if 0 is a solution.

Solution: For any linear operator $\mathcal{L}$ we have that $\mathcal{L}(0) = 0$. Indeed, take any element u from the domain of $\mathcal{L}$. Clearly, $0u = 0$. The linearity of $\mathcal{L}$ implies that $\mathcal{L}(0) = \mathcal{L}(0u) = 0 \cdot \mathcal{L}(u) = 0$.

A linear homogeneous equation has the form $\mathcal{L}(u) = 0$, where $\mathcal{L}$ is a linear operator. By the above 0 is a solution.

A linear equation has the form $\mathcal{L}(u) = f$, where $\mathcal{L}$ is a linear operator and f is given. If 0 is a solution then $\mathcal{L}(0) = f$. But $\mathcal{L}(0) = 0$. Hence $f = 0$ and the equation is homogeneous.

Problem 3. Consider the following equation:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + u^2.$$

(i) Find a nonzero steady-state (independent of t) solution u_0 in the half-plane $x > 0$;

(ii) show that $2u_0$ is not a solution;

(iii) use u_0 to show that the equation is not linear.

Solution: (i) Suppose u_0 is a steady-state solution in the half-plane $x > 0$. Then $u_0(x,t) = v(x)$, where v is a solution of the ODE $v'(x) + (v(x))^2 = 0$ in the half-line $x > 0$. If $v(x) \neq 0$ then $(1/v)'(x) = -v'(x)/(v(x))^2 = 1$. Hence either $v = 0$ or $1/v(x) = x + C$, $C = \text{const}$. In particular, $v(x) = (x + C)^{-1}$ is a nonzero solution in the half-line $x > 0$ for any $C \geq 0$. For example, take $C = 0$. Then $u_0(x,t) = x^{-1}$ is the desired steady-state solution.

(ii) Since u_0 is a steady-state solution, we have that $\frac{\partial u_0}{\partial t} = 0$ and $\frac{\partial u_0}{\partial x} + u_0^2 = 0$. Then $\frac{\partial (2u_0)}{\partial t} = 0$ while $\frac{\partial (2u_0)}{\partial x} + (2u_0)^2 = 2\frac{\partial u_0}{\partial x} + 4u_0^2 = 2u_0^2 \neq 0$. Therefore $2u_0$ is not a solution.

(iii) Assume, on the contrary, that the equation can be transformed into the linear form. Since 0 is obviously a solution, the equation is linear homogeneous (see Problem 2). For a linear homogeneous equation, u_0 is a solution if and only if so is $2u_0$. We have arrived at a contradiction.

Problem 4. Using separation of variables, find a nonzero solution of the equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} - u \qquad (k = \text{const} > 0).$$

Solution: We are looking for a solution of the form $u(x,t) = X(x)T(t)$. Substituting this into the equation, we obtain

$$X(x)T'(t) = kX''(x)T(t) - X(x)T(t),$$
$$\frac{T'(t)}{T(t)} = k\frac{X''(x)}{X(x)} - 1.$$

Since the left-hand side does not depend on x while the right-hand side does not depend on t, it follows that

$$\frac{T'(t)}{T(t)} = k\frac{X''(x)}{X(x)} - 1 = \lambda,$$

where λ is a constant. Then

$$T' = \lambda T, \qquad X'' = k^{-1}(1+\lambda)X.$$

Conversely, if functions T and X are solutions of the above ODEs for the same value of λ, then $u(x,t) = X(x)T(t)$ is a solution of the PDE. For example, we may take $\lambda = -1$, $T(t) = e^{-t}$, and $X(x) = x$. Hence $u(x,t) = e^{-t}x$ is the desired solution.

Problem 5. Determine the eigenvalues λ of the following eigenvalue problem:

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad \phi(0) = 0, \quad \frac{d\phi}{dx}(L) = 0.$$

Analyze three cases: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$. You may assume that the eigenvalues are real.

Solution: $\lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2$, $n = 0, 1, 2, 3, \ldots$

Detailed solution: Case 1: $\lambda > 0$. Here the general solution of the diferential equation is $\phi(x) = C_1 \cos\mu x + C_2 \sin\mu x$, where $\lambda = \mu^2$, $\mu > 0$, and C_1, C_2 are arbitrary constants. Clearly, $\phi(0) = C_1$ and $\phi'(L) = -C_1\mu\sin\mu L + C_2\mu\cos\mu L$. Hence the boundary conditions are satisfied if and only if $C_1 = 0$, $C_2\mu\cos\mu L = 0$. A nonzero solution exists if $\mu L = \pi/2 + n\pi$, $n \in \mathbb{Z}$. That is, if $\mu = (2n+1)\pi/(2L)$, $n \in \mathbb{Z}$. Since $\mu > 0$, we obtain eigenvalues $\lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2$, $n = 0, 1, 2, \ldots$ The corresponding eigenfunctions are $\phi_n(x) = \sin\frac{(2n+1)\pi x}{2L}$.

Case 2: $\lambda = 0$. The general solution of the equation is $\phi(x) = C_1 + C_2 x$, where C_1, C_2 are constants. Since $\phi(0) = C_1$ and $\phi'(L) = C_2$, the boundary value problem has only zero solution. Hence 0 is not an eigenvalue.

Case 3: $\lambda < 0$. The general solution of the equation is $\phi(x) = C_1\cosh\mu x + C_2\sinh\mu x$, where $\lambda = -\mu^2$, $\mu > 0$, and C_1, C_2 are constants. We have that $\phi(0) = C_1$ and $\phi'(L) = C_1\mu\sinh\mu L + C_2\mu\cosh\mu L$. The boundary conditions are satisfied if and only if $C_1 = 0$ and $C_2\mu\cosh\mu L = 0$. Since cosh is a positive function, it follows that the boundary value problem has only zero solution. Hence there are no negative eigenvalues.

Problem 6. Solve the initial-boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0$$

with the following initial and boundary conditions:

(i) $u(x,0) = 6\sin\dfrac{9\pi x}{L}$, $\quad u(0,t) = u(L,t) = 0$;

(ii) $u(x,0) = 3\sin\dfrac{\pi x}{L} - \sin\dfrac{3\pi x}{L}$, $\quad u(0,t) = u(L,t) = 0$;

(iii) $u(x,0) = 6 + 4\cos\dfrac{3\pi x}{L}$, $\quad \dfrac{\partial u}{\partial t}(0,t) = \dfrac{\partial u}{\partial t}(L,t) = 0$;

(iv) $u(x,0) = -3\cos\dfrac{8\pi x}{L}$, $\quad \dfrac{\partial u}{\partial t}(0,t) = \dfrac{\partial u}{\partial t}(L,t) = 0$.

Solution: (i) $u(x,t) = 6\exp\left(-\dfrac{81\pi^2}{L^2}kt\right)\sin\dfrac{9\pi x}{L}$;
(ii) $u(x,t) = 3\exp\left(-\dfrac{\pi^2}{L^2}kt\right)\sin\dfrac{\pi x}{L} - \exp\left(-\dfrac{9\pi^2}{L^2}kt\right)\sin\dfrac{3\pi x}{L}$;
(iii) $u(x,t) = 6 + 4\exp\left(-\dfrac{9\pi^2}{L^2}kt\right)\cos\dfrac{3\pi x}{L}$; (iv) $u(x,t) = -3\exp\left(-\dfrac{64\pi^2}{L^2}kt\right)\cos\dfrac{8\pi x}{L}$.

Detailed solution: The separation of variables provides the following solution. To solve Problems 6(i) and 6(ii), we have to expand the initial data into a series

$$\sum_{n=1}^{\infty} c_n\phi_n,$$

where c_n are constant coefficients and $\phi_n(x) = \sin\frac{n\pi x}{L}$ are eigenfunctions of the eigenvalue problem

$$\phi'' = -\lambda\phi, \qquad \phi(0) = \phi(L) = 0.$$

Then the solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n kt}\phi_n(x),$$

where $\lambda_n = (n\pi/L)^2$ are the corresponding eigenvalues.

To solve Problems 6(iii) and 6(iv), we have to expand the initial data into a series

$$\sum_{n=0}^{\infty} c_n\psi_n,$$

where c_n are constant coefficients, $\psi_0 = 1$ and $\psi_n(x) = \cos\frac{n\pi x}{L}$, $n \geq 1$ are eigenfunctions of the eigenvalue problem

$$\psi'' = -\lambda\psi, \qquad \psi'(0) = \psi'(L) = 0.$$

Then the solution is

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-\lambda_n kt}\psi_n(x),$$

where $\lambda_n = (n\pi/L)^2$ are the corresponding eigenvalues.

In each case, the initial data are already expanded as we need (moreover, the expansion is finite).

Problem 7. Show that all solutions of Problem 6 uniformly approach steady-state solutions as $t \to \infty$.

Solution: Let u_1, u_2, u_3, u_4 be solutions of Problems 6(i), 6(ii), 6(iii), 6(iv), respectively. We have that $|u_1(x,t)| \leq 6\exp\left(-\frac{81\pi^2}{L^2}kt\right)$, $|u_2(x,t)| \leq 3\exp\left(-\frac{\pi^2}{L^2}kt\right) + \exp\left(-\frac{9\pi^2}{L^2}kt\right)$, $|u_3(x,t) - 6| \leq 4\exp\left(-\frac{9\pi^2}{L^2}kt\right)$, $|u_4(x,t)| \leq 3\exp\left(-\frac{64\pi^2}{L^2}kt\right)$.

Hence, as $t \to \infty$, the solutions u_1, u_2, u_4 of the heat equation uniformly approach the steady-state solution $u = 0$ while u_3 approaches the steady-state solution $u = 6$.