# Homework assignment #5

## (due Monday, October 16)

**Problem 1.** Consider the non-Sturm-Liouville differential equation

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + (\lambda\beta(x) + \gamma(x))\phi = 0.$$

Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm-Liouville form:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + (\lambda\sigma(x) + q(x))\phi = 0.$$

Given $\alpha(x)$, $\beta(x)$, and $\gamma(x)$, what are $p(x)$, $\sigma(x)$, and $q(x)$?

**Problem 2.** Consider

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + \alpha u,$$

where $c, \rho, K_0, \alpha$ are functions of $x$, subject to

$$u(0,t) = u(L,t) = 0, \quad u(x,0) = f(x).$$

Assume that the appropriate eigenfunctions are known.

(i) Show that the eigenvalues are positive if $\alpha < 0$.
(ii) Solve the initial value problem.
(iii) Briefly discuss $\lim_{t\to+\infty} u(x,t)$.

**Problem 3.** A Sturm-Liouville problem is called self-adjoint if

$$p\,(uv' - vu')\Big|_a^b = 0$$

for any two functions $u$ and $v$ satisfying the boundary conditions. Show that the following yield self-adjoint problems:

(i) $\phi'(0) = 0$ and $\phi(L) = 0$;
(ii) $\phi'(0) - h\phi(0) = 0$ and $\phi'(L) = 0$.

**Problem 4.** Consider the boundary value problem

$$\phi'' + \lambda\phi = 0 \quad \text{with} \quad \phi(0) - \phi'(0) = 0, \;\; \phi(1) + \phi'(1) = 0.$$

(i) Using the Rayleigh quotient, show that $\lambda \geq 0$. Why is $\lambda > 0$?
(ii) Show that

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

Determine the eigenvalues graphically. Estimate the large eigenvalues.

**Problem 5.** Consider the eigenvalue problem

$$\phi'' + \lambda\phi = 0 \quad \text{with} \quad \phi(0) = \phi'(0) \;\text{ and }\; \phi(1) = \beta\phi'(1).$$

For what values (if any) of $\beta$ is $\lambda = 0$ an eigenvalue?