# Solutions for homework assignment #5

**Problem 1.** Consider the non-Sturm-Liouville differential equation

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + (\lambda\beta(x) + \gamma(x))\phi = 0.$$

Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm-Liouville form:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + (\lambda\sigma(x) + q(x))\phi = 0.$$

Given $\alpha(x)$, $\beta(x)$, and $\gamma(x)$, what are $p(x)$, $\sigma(x)$, and $q(x)$?

**Solution:** $p(x) = e^{A(x)}$, $\sigma(x) = e^{A(x)}\beta(x)$, and $q(x) = e^{A(x)}\gamma(x)$, where $A$ is an anti-derivative of $\alpha$.

**Detailed solution:** The standard Sturm-Liouville equation can be rewritten as

$$p(x)\phi''(x) + p'(x)\phi'(x) + (\lambda\sigma(x) + q(x))\phi = 0$$

(assuming $p$ is differentiable). This is to be the same as the equation

$$H(x)\phi''(x) + H(x)\alpha(x)\phi'(x) + (\lambda H(x)\beta(x) + H(x)\gamma(x))\phi(x) = 0.$$

It follows that $p = H$, $p' = H\alpha$, $\sigma = H\beta$, and $q = H\gamma$. We expect $H$ to be positive. Then the first two relations imply that

$$\frac{p'}{p} = \alpha \quad\Longrightarrow\quad \log p = \int \alpha(x)\,dx \quad\Longrightarrow\quad p = \exp\Big(\int \alpha(x)\,dx\Big).$$

Consequently, $\sigma = p\beta$ and $q = p\gamma$.

**Problem 2.** Consider

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + \alpha u,$$

where $c, \rho, K_0, \alpha$ are functions of $x$, subject to

$$u(0,t) = u(L,t) = 0, \quad u(x,0) = f(x).$$

Assume that the appropriate eigenfunctions are known.

(i) Show that the eigenvalues are positive if $\alpha < 0$.
(ii) Solve the initial value problem.
(iii) Briefly discuss $\lim_{t\to+\infty} u(x,t)$.

**Solution:**

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n}\phi_n(x),$$

where $\lambda_1 < \lambda_2 < \dots$ are eigenvalues of the Sturm-Liouville eigenvalue problem

$$(K_0\phi')' + \alpha\phi + \lambda c\rho\phi = 0, \qquad \phi(0) = \phi(L) = 0,$$

$\phi_1, \phi_2, \ldots$ are the corresponding eigenfunctions, and

$$c_n = \frac{\displaystyle\int_0^L f(x)\phi_n(x)c(x)\rho(x)\,dx}{\displaystyle\int_0^L |\phi_n(x)|^2 c(x)\rho(x)\,dx}.$$

If $\lambda_1 > 0$ then $\lim\limits_{t\to+\infty} u(x,t) = 0$.

**Detailed solution:** We search for the solution of the initial-boundary value problem as a superposition of solutions $u(x,t) = \phi(x)G(t)$ with separated variables of the differential equation that satisfy the boundary conditions.

Substituting $u(x,t) = \phi(x)G(t)$ into the differential equation, we obtain

$$c(x)\rho(x)\phi(x)G'(t) = (K_0(x)\phi'(x))'G(t) + \alpha\phi(x)G(t),$$

$$\frac{G'(t)}{G(t)} = \frac{(K_0(x)\phi'(x))' + \alpha\phi(x)}{c(x)\rho(x)\phi(x)}.$$

Since the left-hand side does not depend on $x$ while the right-hand side does not depend on $t$, it follows that

$$\frac{G'(t)}{G(t)} = \frac{(K_0(x)\phi'(x))' + \alpha\phi(x)}{c(x)\rho(x)\phi(x)} = -\lambda,$$

where $\lambda$ is a constant. Then

$$G' = -\lambda G, \qquad (K_0\phi')' + \alpha\phi + \lambda c\rho\phi = 0.$$

Conversely, if functions $G$ and $\phi$ are solutions of the above ODEs for the same value of $\lambda$, then $u(x,t) = \phi(x)G(t)$ is a solution of the given PDE.

Substituting $u(x,t) = \phi(x)G(t)$ into the boundary conditions, we get

$$\phi(0)G(t) = 0, \qquad \phi(L)G(t) = 0.$$

It is no loss to assume that $G$ is not identically zero. Then the boundary conditions are satisfied if and only if $\phi(0) = \phi(L) = 0$.

To determine $\phi$, we have a regular Sturm-Liouville eigenvalue problem

$$(K_0\phi')' + \alpha\phi + \lambda c\rho\phi = 0, \qquad \phi(0) = \phi(L) = 0.$$

This problem has infinitely many eigenvalues $\lambda_1 < \lambda_2 < \ldots$. Let $\phi_n$ denote an eigenfunction corresponding to the eigenvalue $\lambda_n$. Then $\phi_n$ is determined up to multiplying by a scalar. $\lambda_n$ and $\phi_n$ are related through the Rayleigh quotient

$$\lambda_n = \frac{-K_0\phi_n\phi_n'\Big|_0^L + \displaystyle\int_0^L (K_0|\phi_n'|^2 - \alpha|\phi_n|^2)\,dx}{\displaystyle\int_0^L c\rho|\phi_n|^2\,dx}.$$

Since $\phi_n(0) = \phi_n(L) = 0$, the non-integral term vanishes. If $\alpha < 0$ then it follows that $\lambda_n > 0$.

The function $G$ is to be determined from the equation $G' = -\lambda G$. The general solution of this equation is $G(t) = c_0 e^{-\lambda t}$, where $c_0$ is a constant.

Thus we obtain the following solutions of the heat equation satisfying the boundary conditions:

$$u_n(x,t) = e^{-\lambda_n}\phi_n(x), \quad n = 1, 2, \ldots$$

A superposition of these solutions is a series

$$u(x,t) = \sum\nolimits_{n=1}^{\infty} c_n e^{-\lambda_n} \phi_n(x),$$

where $c_1, c_2, \dots$ are constants. Substituting the series into the initial condition $u(x,0) = f(x)$, we get

$$f(x) = \sum\nolimits_{n=1}^{\infty} c_n \phi_n(x).$$

Hence the initial condition is satisfied if $c_n$ are coefficients of the generalized Fourier series of $f$, that is, if

$$c_n = \frac{\displaystyle\int_0^L f(x)\phi_n(x)c(x)\rho(x)\,dx}{\displaystyle\int_0^L |\phi_n(x)|^2 c(x)\rho(x)\,dx}.$$

If all eigenvalues are positive then the solution $u(x,t)$ vanishes as $t \to +\infty$ because each term of the corresponding series contains a decaying factor $e^{-\lambda_n}$.

**Problem 3.** A Sturm-Liouville problem is called self-adjoint if

$$p\,(uv' - vu')\,\Big|_a^b = 0$$

for any two functions $u$ and $v$ satisfying the boundary conditions. Show that the following yield self-adjoint problems:

(i) $\phi'(0) = 0$ and $\phi(L) = 0$;
(ii) $\phi'(0) - h\phi(0) = 0$ and $\phi'(L) = 0$.

**Solution:** (i) Suppose that functions $u$ and $v$ satisfy the boundary conditions $u'(0) = v'(0) = 0$ and $u(L) = v(L) = 0$. The first condition implies that $u(0)v'(0) - v(0)u'(0) = 0$ while the second one implies that $u(L)v'(L) - v(L)u'(L) = 0$. It follows that

$$p\,(uv' - vu')\,\Big|_0^L = 0.$$

(ii) Suppose that functions $u$ and $v$ satisfy the boundary conditions $u'(0)-hu(0) = v'(0)-hv(0) = 0$ and $u'(L) = v'(L) = 0$. The first condition implies that $u(0)v'(0) - v(0)u'(0) = u(0) \cdot hv(0) - v(0) \cdot hu(0) = 0$ while the second one implies that $u(L)v'(L) - v(L)u'(L) = 0$. It follows that

$$p\,(uv' - vu')\,\Big|_0^L = 0.$$

**Problem 4.** Consider the boundary value problem

$$\phi'' + \lambda\phi = 0 \quad \text{with} \quad \phi(0) - \phi'(0) = 0, \;\; \phi(1) + \phi'(1) = 0.$$

(i) Using the Rayleigh quotient, show that $\lambda \geq 0$. Why is $\lambda > 0$?
(ii) Show that

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

Determine the eigenvalues graphically. Estimate the large eigenvalues.

**Solution:** $\sqrt{\lambda_n} \sim (n-1)\pi$ as $n \to \infty$.

**Detailed solution:** (i) The Rayleigh quotient relates an eigenfunction $\phi$ to the corresponding eigenvalue $\lambda$:

$$\lambda = \frac{-\phi\phi' \Big|_0^1 + \displaystyle\int_0^1 |\phi'(x)|^2\,dx}{\displaystyle\int_0^1 |\phi(x)|^2\,dx}.$$

The boundary conditions imply that

$$-\phi\phi' \Big|_0^1 = -\phi(1)\phi'(1) + \phi(0)\phi'(0) = |\phi(1)|^2 + |\phi(0)|^2 \geq 0.$$

It follows that $\lambda \geq 0$. Furthermore, if $\lambda = 0$ then $\phi(0) = \phi(1) = 0$ and $\phi'$ is identically zero, which is not possible since then $\phi$ is also identically zero. Thus $\lambda > 0$.

(ii) For any $\lambda > 0$ the general solution of the equation $\phi'' + \lambda\phi = 0$ is $\phi(x) = c_1 \cos\mu x + c_2 \sin\mu x$, where $\mu = \sqrt{\lambda}$ and $c_1, c_2$ are constants. Substituting this into the boundary conditions $\phi(0) = \phi'(0)$ and $\phi(1) = -\phi'(1)$, we obtain

$$c_1 = c_2\mu, \qquad c_1\cos\mu + c_2\sin\mu = c_1\mu\sin\mu - c_2\mu\cos\mu.$$

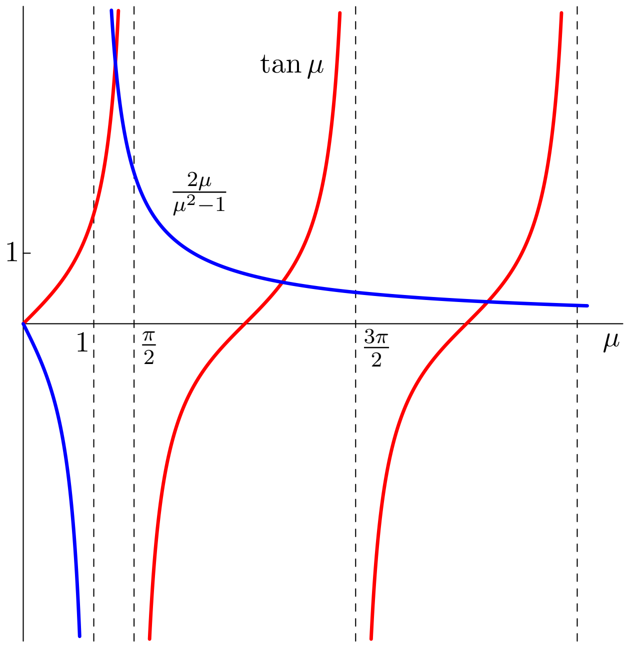
It follows that

$$c_2(\mu\cos\mu + \sin\mu) = c_2(\mu^2\sin\mu - \mu\cos\mu),$$
$$c_2((\mu^2-1)\sin\mu - 2\mu\cos\mu) = 0.$$

A nonzero solution of the boundary value problem exists if and only if $(\mu^2-1)\sin\mu = 2\mu\cos\mu$. Since $\cos 1 \neq 0$, the latter equality implies that $\mu^2 \neq 1$ and $\cos\mu \neq 0$. Hence it is equivalent to

$$\tan\mu = \frac{2\mu}{\mu^2-1}.$$

It remains to recall that $\mu = \sqrt{\lambda}$.



Let $0 < \lambda_1 < \lambda_2 < \dots$ be the eigenvalues. Graphically, we establish that $1 < \sqrt{\lambda_1} < \pi/2$, $\pi/2 < \sqrt{\lambda_2} < 3\pi/2$, and $(2n-3)\pi/2 < \sqrt{\lambda_n} < (2n-1)\pi/2$ for $n = 3, 4, \dots$. Moreover, $\sqrt{\lambda_n} \approx (n-1)\pi$ for a large $n$.

**Problem 5.** Consider the eigenvalue problem

$$\phi'' + \lambda\phi = 0 \quad \text{with} \quad \phi(0) = \phi'(0) \ \text{ and } \ \phi(1) = \beta\phi'(1).$$

For what values (if any) of $\beta$ is $\lambda = 0$ an eigenvalue?

**Solution:** $\beta = 2$.

**Detailed solution:** In the case $\lambda = 0$, the general solution of the equation $\phi'' + \lambda\phi = 0$ is a linear function $\phi(x) = c_1 + c_2 x$, where $c_1, c_2$ are constants. Substituting it into the boundary conditions $\phi(0) = \phi'(0)$ and $\phi(1) = \beta\phi'(1)$, we obtain equalities $c_1 = c_2$, $c_1 + c_2 = \beta c_2$. They imply that $2c_1 = 2c_2 = \beta c_2$. If $\beta \neq 2$, it follows that $c_1 = c_2 = 0$, hence there are no eigenfunctions with eigenvalue $\lambda = 0$. If $\beta = 2$ then $\phi(x) = 1 + x$ is indeed an eigenfunction.