Homework assignment #7

(due Friday, November 3)

Problem 1. Consider the two-dimensional eigenvalue problem with $\sigma > 0$

$$\nabla^2\phi + \lambda\sigma(x,y)\phi = 0 \quad \text{in the domain } D,$$
$$\phi = 0 \quad \text{on the boundary } \partial D.$$

(i) Prove that $\lambda \geq 0$.
(ii) Is $\lambda = 0$ an eigenvalue, and if so, what is the eigenfunction?

Hint: Derive the Rayleigh quotient for this particular problem.

Problem 2. Solve as simply as possible an initial-boundary value problem for the wave equation in a circle:

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\nabla^2 u$$

with $u(a,\theta,t) = 0$, $u(r,\theta,0) = 0$, and $\dfrac{\partial u}{\partial t}(r,\theta,0) = \alpha(r)\sin 3\theta$.

Problem 3. Consider a vibrating quarter-circular membrane, $0 < r < a$, $0 < \theta < \pi/2$, with $u = 0$ on the entire boundary.

(i) Determine an expression for the frequencies of vibration.
(ii) Solve the initial value problem if $u(r,\theta,0) = g(r,\theta)$, $\dfrac{\partial u}{\partial t}(r,\theta,0) = 0$.

Problem 4. Consider Bessel's differential equation

$$z^2\frac{d^2 f}{dz^2} + z\frac{df}{dz} + (z^2 - m^2)f = 0.$$

(i) Let $f = y/z^{1/2}$. Derive that

$$\frac{d^2 y}{dz^2} + \left(1 + \frac{1}{4}z^{-2} - m^2 z^{-2}\right) y = 0.$$

(ii) Using (i), determine exact expressions for $J_{1/2}(z)$ and $Y_{1/2}(z)$. Use and verify the asymptotics as $z \to 0$ and as $z \to \infty$.

Problem 5. Solve Laplace's equation inside a circular cylinder subject to the boundary conditions (in the cylindrical coordinates r, θ, z)

$$u(r,\theta,0) = 0, \qquad u(r,\theta,H) = \beta(r)\cos 3\theta, \qquad \frac{\partial u}{\partial r}(a,\theta,z) = 0.$$

Under what condition does a solution exist?