Math 412-501

Theory of Partial Differential Equations

Lecture 11: Review for Exam 1.

PDEs: two variables

heat equation: $$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$$

wave equation: $$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}$$

Laplace's equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

These equations are **linear homogeneous**.

PDEs: three variables

heat equation: $$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

wave equation: $$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Laplace's equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

One-dimensional heat equation

Describes heat conduction in a rod:

$$\boxed{c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q}$$

$K_0 = K_0(x)$, $c = c(x)$, $\rho = \rho(x)$, $Q = Q(x, t)$.

Assuming K_0, c, ρ are constant (uniform rod) and $Q = 0$ (no heat sources), we obtain

$$\boxed{\frac{\partial u}{\partial t} = k\, \frac{\partial^2 u}{\partial x^2}}$$

where $k = K_0 (c\rho)^{-1}$.

One-dimensional wave equation

Describes vibrations of a perfectly elastic string:

$$\boxed{\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + \rho(x)Q(x,t)}$$

Assuming $\rho = \text{const}$ and $Q = 0$, we obtain

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}}$$

where $c^2 = T_0/\rho$.

Initial-boundary value problem

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,\;\; 0 \le t \le T.$$

Initial condition: $u(x,0) = f(x)$, where $f : [0, L] \to \mathbb{R}$.

Boundary conditions: $u(0,t) = u_1(t)$, $\frac{\partial u}{\partial x}(L,t) = \phi_2(t)$, where $u_1, \phi_2 : [0, T] \to \mathbb{R}$.

Initial-boundary value problem = PDE + initial condition(s) + boundary conditions

D'Alembert's solution of 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \ \ -\infty < t < \infty$$

Change of independent variables:

$$w = x + ct, \qquad z = x - ct.$$

Wave equation in new coordinates: $\dfrac{\partial^2 u}{\partial w \, \partial z} = 0.$

General solution: $\quad u(w, z) = B(z) + C(w),$

where $B, C : \mathbb{R} \to \mathbb{R}$ are arbitrary functions.

General solution of the 1D wave equation:

$$\boxed{u(x, t) = B(x - ct) + C(x + ct)}$$

Initial value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x, t < \infty,$$

$$u(x,0) = f(x), \;\; \frac{\partial u}{\partial t}(x,0) = g(x), \;\; -\infty < x < \infty.$$

General solution: $u(x,t) = B(x - ct) + C(x + ct)$.

We substitute it into initial conditions:

$B(x) + C(x) = f(x), \qquad -cB'(x) + cC'(x) = g(x).$

Unknown functions B and C can be found from these equations.

The initial value problem has a unique solution:

$$u(x,t) = \frac{1}{2}\Big(f(x-ct) + f(x+ct) + G(x+ct) - G(x-ct)\Big)$$

where G is an arbitrary anti-derivative of g/c.

Another representation of this solution:

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

(d'Alembert's formula)

Semi-infinite string

Initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad x \geq 0;$$

$$u(x,0) = f(x), \;\; \frac{\partial u}{\partial t}(x,0) = g(x), \;\; x \geq 0;$$

$$u(0,t) = 0 \;\; \text{(fixed end)}.$$

General solution: $u(x,t) = B(x - ct) + C(x + ct)$.

We substitute it into initial and boundary conditions:

$B(x) + C(x) = f(x), \quad -cB'(x) + cC'(x) = g(x),$

$x \geq 0; \qquad B(-ct) + C(ct) = 0.$

Unknown functions B and C can be found from these equations.

Another approach

Initial-boundary value problem has a **unique** solution and this solution can be extended to the whole plane.

Hence the problem can be solved as follows:

- extend f and g to the whole line *somehow*;
- solve the initial value problem in the whole plane;
- if the boundary condition holds, we are done!

Hints on how to satisfy the boundary condition:

- the boundary condition $u(0,t) = 0$ (fixed end) holds if the (extended) functions f and g are **odd**;
- The boundary condition $\frac{\partial u}{\partial x}(0,t) = 0$ (free end) holds if the (extended) functions f and g are **even**.

Separation of variables

The method applies to certain linear PDEs, for example, heat equation, wave equation, Laplace's equation.

Basic idea: to find a solution of the PDE (function of many variables) as the product of several functions, each depending only on one variable.

For example, $u(x, t) = B(x)C(t)$.

Heat equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$$

Suppose $u(x,t) = \phi(x)G(t)$. Then

$$\frac{\partial u}{\partial t} = \phi(x)\frac{dG}{dt}, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{d^2\phi}{dx^2}G(t).$$

Hence

$$\phi(x)\frac{dG}{dt} = k\,\frac{d^2\phi}{dx^2}G(t).$$

Divide both sides by $k \cdot \phi(x) \cdot G(t) = k \cdot u(x,t)$:

$$\frac{1}{kG} \cdot \frac{dG}{dt} = \frac{1}{\phi} \cdot \frac{d^2\phi}{dx^2}.$$

It follows that

$$\frac{1}{kG} \cdot \frac{dG}{dt} = \frac{1}{\phi} \cdot \frac{d^2\phi}{dx^2} = -\lambda = \text{const.}$$

λ is called the **separation constant**. The variables have been separated:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi,$$
$$\frac{dG}{dt} = -\lambda kG.$$

Proposition Suppose ϕ and G are solutions of the above ODEs for the same value of λ. Then $u(x,t) = \phi(x)G(t)$ is a solution of the heat equation.

Boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$
$$u(0,t) = u(L,t) = 0.$$

We are looking for solutions $u(x,t) = \phi(x)G(t)$.

PDE holds if

$$\frac{d^2\phi}{dx^2} = -\lambda\phi,$$
$$\frac{dG}{dt} = -\lambda k G$$

for the same constant λ.

Boundary conditions hold if

$$\phi(0) = \phi(L) = 0.$$

Boundary value problem:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \qquad 0 \le x \le L,$$
$$\phi(0) = \phi(L) = 0.$$

There is an obvious solution: 0.
When is it **not unique?**

If for some value of λ the boundary value problem has a nonzero solution ϕ, then this λ is called an **eigenvalue** and ϕ is called an **eigenfunction**.

The **eigenvalue problem** is to find all eigenvalues (and corresponding eigenfunctions).

Eigenvalue problem

$$\phi'' = -\lambda\phi, \qquad \phi(0) = \phi(L) = 0.$$

We are looking only for real eigenvalues.

Three cases: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

Case 1: $\lambda > 0$. $\quad \phi(x) = C_1 \cos \mu x + C_2 \sin \mu x$,
where $\lambda = \mu^2$, $\mu > 0$.

$\phi(0) = \phi(L) = 0 \implies C_1 = 0, \;\; C_2 \sin \mu L = 0.$

A nonzero solution exists if $\mu L = n\pi$, $n \in \mathbb{Z}$.

So $\lambda_n = (\frac{n\pi}{L})^2$, $n = 1, 2, \dots$ are eigenvalues and $\phi_n(x) = \sin \frac{n\pi x}{L}$ are corresponding eigenfunctions.

Separation of variables: summary

Eigenvalue problem: $\phi'' = -\lambda\phi$, $\phi(0) = \phi(L) = 0$.

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2$, $n = 1, 2, \ldots$

Eigenfunctions: $\phi_n(x) = \sin\frac{n\pi x}{L}$.

Solution of the heat equation: $u(x,t) = \phi(x)G(t)$.

$$\frac{dG}{dt} = -\lambda kG \implies G(t) = C_0 \exp(-\lambda kt)$$

Theorem For $n = 1, 2, \ldots$, the function

$$\boxed{u(x,t) = e^{-\lambda_n kt}\phi_n(x) = \exp(-\frac{n^2\pi^2}{L^2}kt)\sin\frac{n\pi x}{L}}$$

is a solution of the following boundary value problem for the heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(L,t) = 0.$$

How do we solve the initial-boundary value problem?

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$

$$u(x,0) = f(x), \quad u(0,t) = u(L,t) = 0.$$

- Expand the function f into a series

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}.$$

- Write the solution:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left(-\frac{n^2\pi^2}{L^2}kt\right) \sin\frac{n\pi x}{L}.$$

(Fourier's solution)

Fourier's solution (insulated ends)

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0.$$

- Expand the function f into a series

$$f(x) = A_0 + \sum\nolimits_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}.$$

- Write the solution:

$$u(x,t) = A_0 + \sum\nolimits_{n=1}^{\infty} A_n \exp\Big(-\frac{n^2\pi^2}{L^2}kt\Big)\cos\frac{n\pi x}{L}.$$

Fourier's solution (circular ring)

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad -L \le x \le L,$$

$$u(x,0) = f(t),$$
$$u(-L,t) = u(L,t), \quad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t).$$

- Expand the function f into a series

$$f(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right).$$

- Write the solution:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} \exp\left(-\tfrac{n^2\pi^2}{L^2}kt\right)\left(A_n \cos\tfrac{n\pi x}{L} + B_n \sin\tfrac{n\pi x}{L}\right).$$

Fourier series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

To each integrable function $f : [-L, L] \to \mathbb{R}$ we associate a Fourier series such that

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

and for $n \geq 1$,

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx.$$

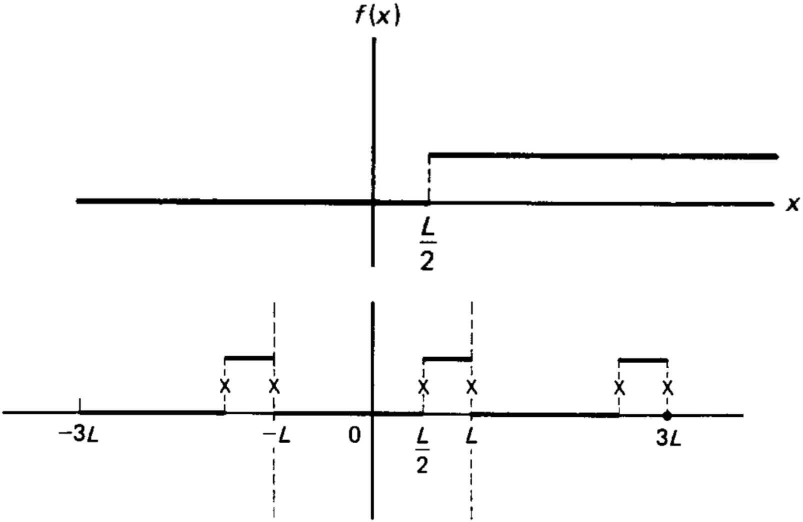
Convergence theorem

Suppose $f : [-L, L] \to \mathbb{R}$ is a **piecewise smooth** function.

Let $F : \mathbb{R} \to \mathbb{R}$ be the $2L$-**periodic extension** of f.

Theorem The Fourier series of the function f converges everywhere. The sum at a point x is equal to $F(x)$ if F is continuous at x. Otherwise the sum is equal to

$$\frac{F(x-) + F(x+)}{2}.$$



Function and its Fourier series

Fourier sine and cosine series

Suppose $f(x)$ is an integrable function on $[0, L]$.
The Fourier sine series of f

$$\sum_{n=1}^{\infty} B_n \sin \tfrac{n\pi x}{L}$$

and the Fourier cosine series of f

$$A_0 + \sum_{n=1}^{\infty} A_n \cos \tfrac{n\pi x}{L}$$

are defined as follows:

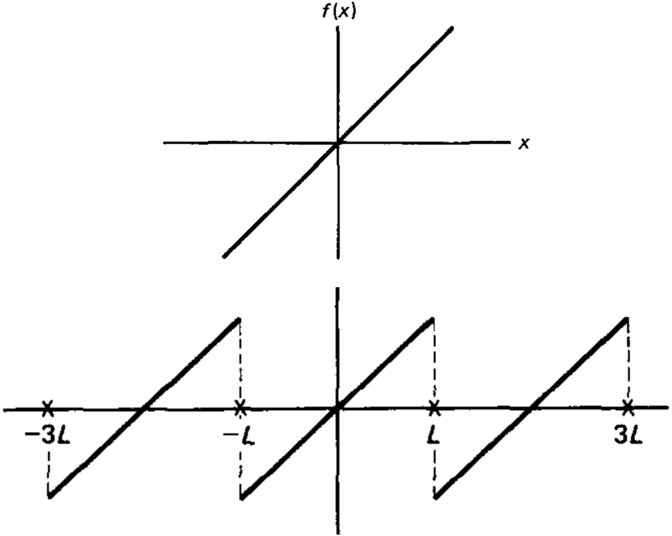
$$B_n = \tfrac{2}{L} \int_0^L f(x) \sin \tfrac{n\pi x}{L} \, dx;$$

$$A_0 = \tfrac{1}{L} \int_0^L f(x) \, dx, \qquad A_n = \tfrac{2}{L} \int_0^L f(x) \cos \tfrac{n\pi x}{L} \, dx, \quad n \geq 1.$$

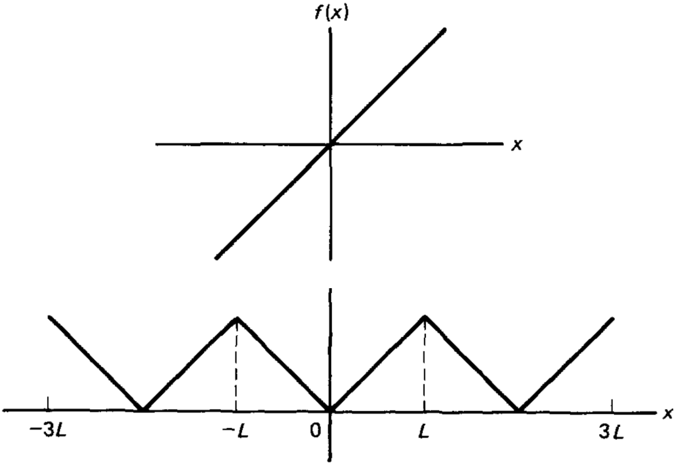
Convergence Theorem If a function $f : [0, L] \to \mathbb{R}$ is piecewise smooth then both Fourier sine and Fourier cosine series of f converge to $f(x)$ at any point $0 < x < L$ of continuity.

Proposition (i) The Fourier series of a function $f : [-L, L] \to \mathbb{R}$ contains only sines if the function is odd.

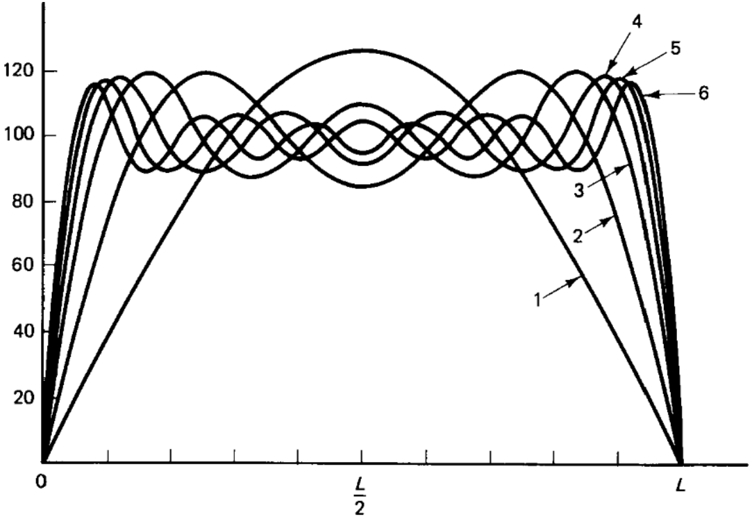
(ii) The Fourier series of a function $f : [-L, L] \to \mathbb{R}$ contains only a constant and cosines if the function is even.



Fourier sine series of $f(x) = x$



Fourier cosine series of $f(x) = x$



$p_n(x), \quad 1 \leq n \leq 6.$

Gibbs' phenomenon

The partial sum $p_n(x)$ attains its maximal value v_n on the interval $0 \leq x \leq L$ at two points x_n^+, x_n^- such that $x_n^+ \to L$ and $x_n^- \to 0$ as $n \to \infty$.

Actually, $x_n^- = \frac{L}{2n}$, $x_n^+ = L - \frac{L}{2n}$.

The maximal **overshoot** $v_n = p_n(x_n^\pm)$ satisfies $v_1 > v_2 > v_3 > \dots$ and $\lim\limits_{n\to\infty} v_n = v_\infty > \mathbf{100}$.

$$\text{Actually, } v_\infty = \frac{200}{\pi}\int_0^\pi \frac{\sin y}{y}\,dy \approx 117.898$$

The **Gibbs phenomenon** occurs for any piecewise smooth function at any discontinuity. The ultimate overshoot rate of $\approx 9\%$ of the jump is universal.

Example

$f(x) = e^x$.

Find the Fourier cosine series ($0 \le x \le L$).

$$A_0 = \frac{1}{L}\int_0^L e^x\,dx.$$

For $n \ge 1$,

$$A_n = \frac{2}{L}\int_0^L e^x \cos\frac{n\pi x}{L}\,dx.$$

Table of integrals:

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}.$$