Math 412-501

Theory of Partial Differential Equations

Lecture 1: Introduction. Heat equation

Definitions

A **differential equation** is an equation involving an unknown function and certain of its derivatives.

An **ordinary differential equation (ODE)** is an equation involving an unknown function of one variable and certain of its derivatives.

A **partial differential equation (PDE)** is an equation involving an unknown function of two or more variables and certain of its partial derivatives.

Examples

$x^2 + 2x + 1 = 0$	(algebraic equation)
$f(2x) = 2(f(x))^2 - 1$	(functional equation)
$f'(t) + t^2 f(t) = 4$	(ODE)
$\frac{\partial u}{\partial x} + 3\frac{\partial^2 u}{\partial x \partial y} - u\frac{\partial u}{\partial y}$	(not an equation)
$\frac{\partial u}{\partial x} - 5\frac{\partial u}{\partial y} = u$	(PDE)
$u + u^2 = \frac{\partial^2 u}{\partial x \partial y}(0, 0)$	(functional-differential equation)

heat equation: $$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$$

wave equation: $$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}$$

Laplace's equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

In the first two equations, $u = u(x, t)$. In the latter one, $u = u(x, y)$.

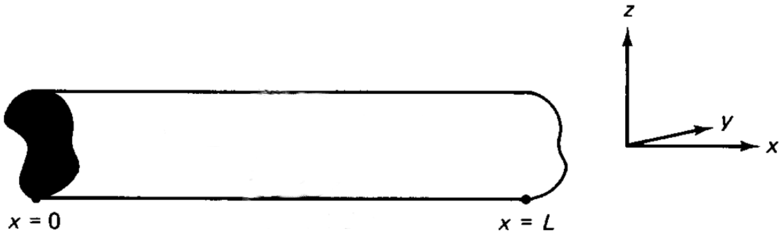
heat equation:
$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

wave equation:
$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

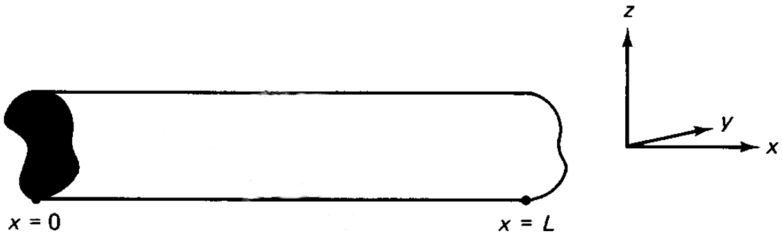
Laplace's equation:
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

In the first two equations, $u = u(x, y, t)$. In the latter one, $u = u(x, y, z)$.

Heat conduction in a rod



$u(x,t)$ = temperature
$e(x,t)$ = thermal energy density (thermal energy per unit volume)
$Q(x,t)$ = density of heat sources (heat energy per unit volume generated per unit time)



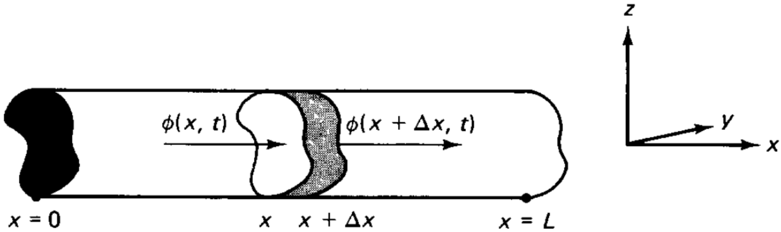
$\phi(x,t) =$ heat flux (thermal energy flowing per unit surface per unit time)

$\phi(x,t) > 0$ if heat energy is flowing to the right,
$\phi(x,t) < 0$ if heat energy is flowing to the left

Conservation of heat energy (in a volume in a period of time):

change of heat energy	$=$	heat energy flowing across boundary	$+$	heat energy generated inside

rate of change of heat energy	$=$	heat energy flowing across boundary per unit time	$+$	heat energy generated inside per unit time



$A =$ area of a section

heat energy $= e(x,t) \cdot A \cdot \Delta x$

rate of change of heat energy $= \dfrac{\partial}{\partial t}\Big(e(x,t) \cdot A \cdot \Delta x\Big)$

heat energy flowing across boundary per unit time

$= \phi(x,t) \cdot A - \phi(x+\Delta x, t) \cdot A$

heat energy generated inside per unit time

$= Q(x,t) \cdot A \cdot \Delta x$

$$\frac{\partial}{\partial t}\Big(e(x,t)\cdot A\cdot \Delta x\Big) = \phi(x,t)\cdot A - \phi(x+\Delta x,t)\cdot A + Q(x,t)\cdot A\cdot \Delta x$$

$$\frac{\partial e(x,t)}{\partial t} = \frac{\phi(x,t)-\phi(x+\Delta x,t)}{\Delta x} + Q(x,t)$$

$$\boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q}$$

$c(x) =$ specific heat or heat capacity (the heat energy supplied to a unit mass of a substance to raise its temperature one unit)

$\rho(x) =$ mass density (mass per unit volume)

Thermal energy in a volume is equal to the energy it takes to raise the temperature of the volume from a reference temperature (zero) to its actual temperature.

$$e(x,t) \cdot A \cdot \Delta x = c(x) u(x,t) \cdot \rho(x) \cdot A \cdot \Delta x$$

$$e(x,t) = c(x)\rho(x)u(x,t)$$

$$\boxed{c\rho\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q}$$

Fourier's law of heat conduction:

$$\phi = -K_0\frac{\partial u}{\partial x},$$

where $K_0 = K_0(x,u)$ is called the *thermal conductivity*.

Heat equation:

$$\boxed{c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q}$$

Assuming $K_0 = \text{const}$, we have

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + Q$$

Assuming $K_0 = \text{const}$, $c = \text{const}$, $\rho = \text{const}$ (uniform rod), and $Q = 0$ (no heat sources), we obtain

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2},$$

where $k = K_0(c\rho)^{-1}$ is called the *thermal diffusivity*.