Math 412-501

Theory of Partial Differential Equations

Lecture 2: Diffusion equation. Wave equation. Boundary conditions.

heat equation: $$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$$

wave equation: $$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}$$

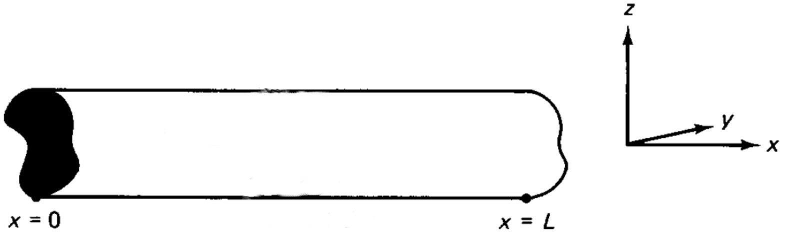
Laplace's equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

heat equation: $$\frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

wave equation: $$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Laplace's equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Heat conduction in a rod



$u(x, t) = \text{temperature}$

Heat equation: $$\boxed{c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q}$$

$K_0 = K_0(x)$, $c = c(x)$, $\rho = \rho(x)$, $Q = Q(x, t)$.

Assuming K_0, c, ρ are constant (uniform rod) and $Q = 0$ (no heat sources), we obtain

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2},$$

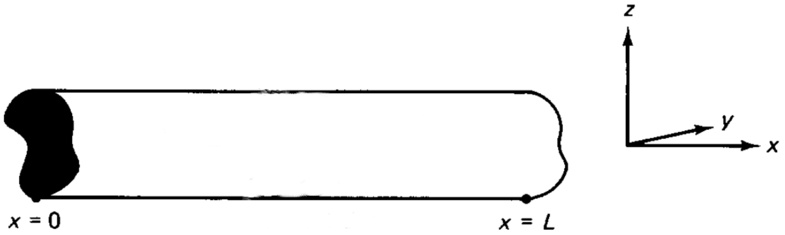
where $k = K_0(c\rho)^{-1}$ is called the *thermal diffusivity*.

Heat equation is derived from two physical laws:

- conservation of heat energy,
- Fourier's low of heat conduction.

The heat equation is also called the **diffusion equation**.

Pollutant diffusion in a tube



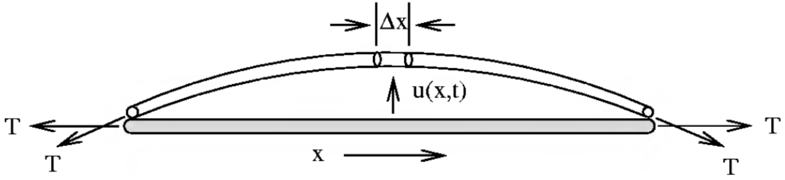
$u(x,t) =$ concentration of the chemical

- conservation of mass
- Fick's law of diffusion

$$\boxed{\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}}$$

$k =$ chemical diffusivity

Vibration of a stretched string

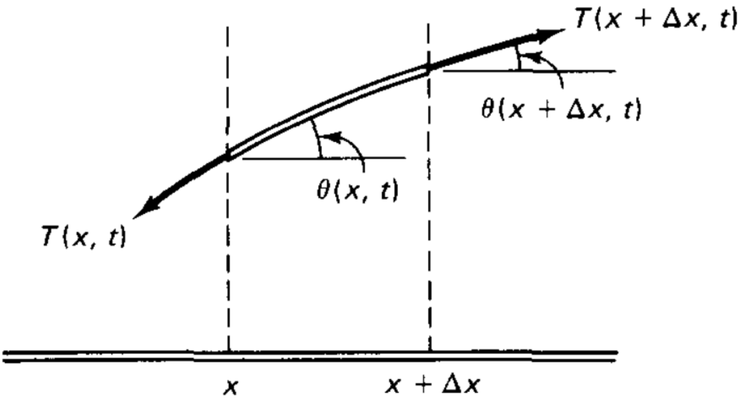


$u(x, t)$ = vertical displacement

Newton's law: mass $\times$ acceleration = force

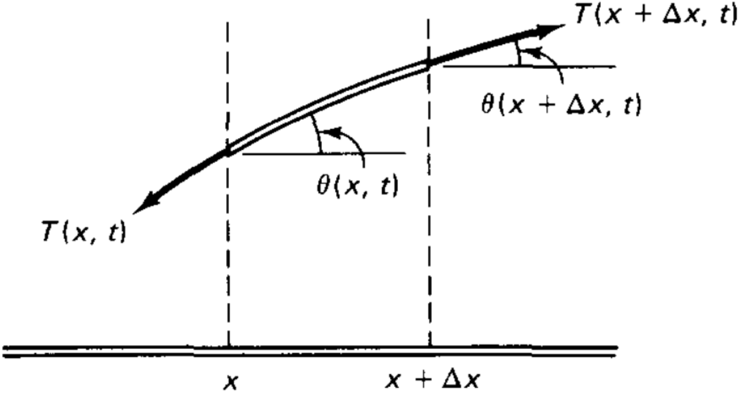
$\rho(x)$ = mass density

$T(x, t)$ = magnitude of tensile force
$Q(x, t)$ = (vertical) external forces on a unit mass

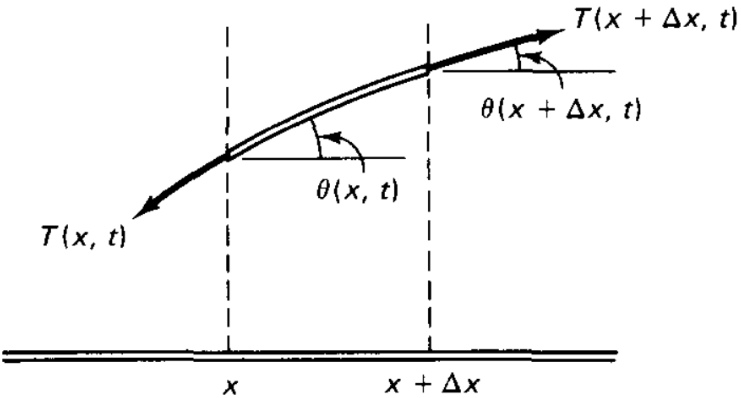


perfectly flexible string: no resistance to bending

$\theta(x,t)$ = angle between the horizon and the string



$$\tan\theta = \frac{\partial u}{\partial x}$$



vertical component of tensile force $=$

$$T(x+\Delta x, t)\cdot \sin\theta(x+\Delta x, t) - T(x,t)\cdot \sin\theta(x,t)$$

$$\rho(x) \cdot \Delta x \cdot \frac{\partial^2 u}{\partial t^2} = T(x + \Delta x, t) \cdot \sin\theta(x + \Delta x, t)$$

$$- T(x, t) \cdot \sin\theta(x, t) + \rho(x) \cdot \Delta x \cdot Q(x, t)$$

$$\rho(x) \cdot \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\Big(T(x, t) \cdot \sin\theta(x, t)\Big) + \rho(x) \cdot Q(x, t)$$

We assume that $\theta << 1$, hence $\sin\theta \approx \tan\theta$.

$$\boxed{\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\Big(T\frac{\partial u}{\partial x}\Big) + \rho(x)Q(x, t)}$$

perfectly elastic string: tension is proportional to stretching (Hooke's law)

Since $\theta << 1$, we assume $T(x,t) \approx T_0 = \text{const}$.

$$\boxed{\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + \rho(x)Q(x,t)}$$

Assuming $\rho = \text{const}$ and $Q = 0$, we obtain

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}}$$

where $c^2 = T_0/\rho$.

This is **one-dimensional wave equation**.

Initial and boundary conditions for ODEs

$y'(t) = y(t)$, $0 \leq t \leq L$.

General solution: $y(t) = C_1 e^t$, where $C_1 = \text{const}$.

To determine a unique solution, we need one **initial condition**.

For example, $y(0) = 1$. Then $y(t) = e^t$ is the unique solution.

$y''(t) = -y(t)$, $0 \leq t \leq L$.
General solution: $y(t) = C_1 \cos t + C_2 \sin t$, where C_1, C_2 are constant.

To determine a unique solution, we need **two** initial conditions. For example, $y(0) = 1$, $y'(0) = 0$. Then $y(t) = \cos t$ is the unique solution.

Alternatively, we may impose boundary conditions. For example, $y(0) = 0$, $y(L) = 1$. In the case $L = \pi/2$, $y(t) = \sin t$ is the unique solution.

Initial value problem = ODE + initial conditions
Boundary value problem = ODE + boundary conditions

Initial value problem $y'' = -y$, $y(0) = a$, $y'(0) = b$ always has a unique solution.

Boundary value problem $y'' = -y$, $y(0) = a$, $y(L) = b$ may not have a unique solution for some triples (a, b, L).

For example, let $L = \pi$ and $a = 0$. Then the boundary value problem has no solution if $b \neq 0$. In the case $b = 0$, it has infinitely many solutions $y(t) = C_1 \sin t$, $C_1 = \text{const}$.

Heat equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,\ 0 \le t \le T.$$

Initial condition: $u(x,0) = f(x)$, where $f : [0, L] \to \mathbb{R}$.

Boundary conditions: $u(0,t) = u_1(t)$, $u(L,t) = u_2(t)$, where $u_1, u_2 : [0, T] \to \mathbb{R}$.

Boundary conditions of the **first kind**: prescribed temperature.

Another boundary conditions: $\dfrac{\partial u}{\partial x}(0, t) = \phi_1(t)$, $\dfrac{\partial u}{\partial x}(L, t) = \phi_2(t)$, where $\phi_1, \phi_2 : [0, T] \to \mathbb{R}$.

Boundary conditions of the **second kind**: prescribed heat flux.

A particular case: $\dfrac{\partial u}{\partial x}(0, t) = \dfrac{\partial u}{\partial x}(L, t) = 0$ (insulated boundary).

Robin conditions:

$$-\frac{\partial u}{\partial x}(0,t) = -h \cdot \Big(u(0,t) - u_1(t)\Big),$$

$$-\frac{\partial u}{\partial x}(L,t) = h \cdot \Big(u(L,t) - u_2(t)\Big),$$

where $h = \text{const} > 0$ and $u_1, u_2 : [0,T] \to \mathbb{R}$.

Boundary conditions of the **third kind**: Newton's law of cooling.

Also, we may consider **mixed** boundary conditions, for example, $u(0,t) = u_1(t)$, $\dfrac{\partial u}{\partial x}(L,t) = \phi_2(t)$.

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq L,\ 0 \leq t \leq T.$$

Two initial conditions: $u(x,0) = f(x)$,
$\frac{\partial u}{\partial t}(x,0) = g(x)$, where $f, g : [0,L] \to \mathbb{R}$.

Some boundary conditions: $u(0,t) = u(L,t) = 0$.

Dirichlet conditions: fixed ends.

Another boundary conditions:
$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$.
Neumann conditions: free ends.