Math 412-501

Theory of Partial Differential Equations

Lecture 3:
Steady-state solutions of the heat equation.
D'Alembert's solution of the wave equation.

One-dimensional heat equation

$$\boxed{c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q}$$

$$K_0 = K_0(x),\ c = c(x),\ \rho = \rho(x),\ Q = Q(x,t).$$

Assuming K_0, c, ρ are constant (uniform rod) and $Q = 0$ (no heat sources), we obtain

$$\boxed{\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}}$$

where $k = K_0(c\rho)^{-1}$.

Initial-boundary value problem

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,\ \ 0 \le t \le T.$$

Initial condition: $u(x,0) = f(x)$, where $f : [0, L] \to \mathbb{R}$.

Boundary conditions: $u(0,t) = u_1(t)$, $\frac{\partial u}{\partial x}(L,t) = \phi_2(t)$, where $u_1, \phi_2 : [0, T] \to \mathbb{R}$.

Initial-boundary value problem = PDE + initial condition(s) + boundary conditions

Steady-state solutions

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q, \quad 0 \le x \le L, \;\; 0 \le t < \infty$$

A solution u of the heat equation is called an **equilibrium** (or **steady-state**) solution if it does not depend on time, that is, $u(x, t_1) = u(x, t_2)$ for any $0 \le x \le L$ and $0 \le t_1 < t_2$.

Hence $u(x, t) = v(x)$, where $v : [0, L] \to \mathbb{R}$.

In particular, $\dfrac{\partial u}{\partial t} = 0$. Also, $\dfrac{\partial u}{\partial x}(x, t) = \dfrac{dv}{dx}(x)$.

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q, \quad 0 \le x \le L,\ 0 \le t < \infty$$

If a steady-state solution exists, then Q does not depend on time.

Suppose $u(x,t) = v(x)$ is a steady-state solution, then

$$\frac{d}{dx}\left(K_0\frac{dv}{dx}\right) + Q = 0, \quad 0 \le x \le L$$

If a steady-state solution satisfies a boundary condition of the first or second kind, then the boundary condition is time-independent.

$$u(0,t) = u_1(t) \implies u_1 = \text{const}$$
$$\frac{\partial u}{\partial x}(0,t) = \phi_1(t) \implies \phi_1 = \text{const}$$

This is not always so for boundary conditions of the third kind. For example, if $u(0,t) = u_0 = \text{const}$ and $\frac{\partial u}{\partial x}(0,t) = 0$, then the boundary condition

$$\frac{\partial u}{\partial x}(0,t) = h(t)\Big(u(0,t) - u_0\Big)$$

is satisfied for an arbitrary function h.

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q, \quad 0 \le x \le L, \; 0 \le t < \infty$$

Conjecture Assume that boundary conditions are time-independent and there exists a steady-state solution satisfying them. Then an arbitrary solution $u(x,t)$ of the initial-boundary value problem (uniformly) approaches a steady-state solution as $t \to \infty$.

$$\lim_{t \to \infty} u(x,t) = u_\infty(x)$$

$$\frac{d}{dx}\left(K_0 \frac{du_\infty}{dx}\right) + Q = 0, \quad 0 \le x \le L$$

$$\frac{d}{dx}\left(K_0\frac{du}{dx}\right)+Q=0, \quad 0\le x\le L$$

$$(K_0u')'+Q=0$$

$$\int_0^x (K_0u')'(\xi)\,d\xi = -\int_0^x Q(\xi)\,d\xi$$

$$K_0(x)u'(x)-K_0(0)u'(0) = -\int_0^x Q(\xi)\,d\xi$$

$$u'(x)=\frac{1}{K_0(x)}\left(K_0(0)u'(0)-\int_0^x Q(\xi)\,d\xi\right)$$

$$u(x)=u(0)+\int_0^x\left(\frac{K_0(0)u'(0)}{K_0(\eta)}-\frac{1}{K_0(\eta)}\int_0^\eta Q(\xi)\,d\xi\right)d\eta$$

Initial value problem

$$(K_0 u')' + Q = 0, \quad u(0) = C_0,\ u'(0) = C_1$$

has a unique solution

$$u(x) = C_0 + \int_0^x \left(\frac{K_0(0) C_1}{K_0(\eta)} - \frac{1}{K_0(\eta)} \int_0^\eta Q(\xi)\, d\xi \right) d\eta$$

Assuming $K_0 = \text{const}$, we have

$$u(x) = C_0 + C_1 x - \int_0^x \left(\frac{1}{K_0} \int_0^\eta Q(\xi)\, d\xi \right) d\eta$$

Assuming $K_0 = \text{const}$ and $Q = 0$, we have

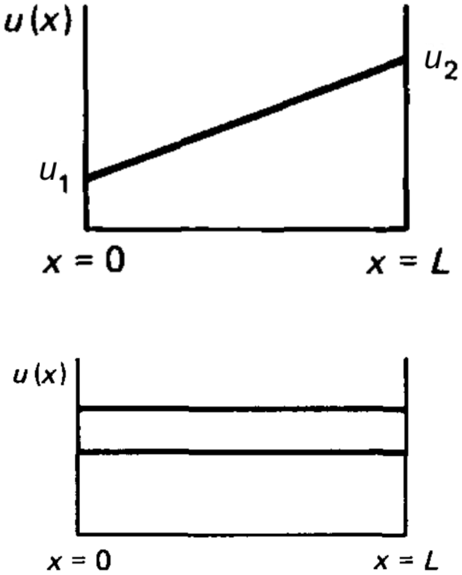
$$u(x) = C_0 + C_1 x$$

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L,\ \ 0 \le t < \infty$$

General steady-state solution: $u(x,t) = C_0 + C_1 x$, where C_0, C_1 are constant.

Boundary conditions: $u(0,t) = u_1,\ u(L,t) = u_2$.
$C_0 = u_1,\ C_0 + C_1 L = u_2 \implies u(x,t) = u_1 + \frac{u_2 - u_1}{L}x$
(unique equilibrium)

Boundary conditions: $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$.
$C_1 = 0 \implies u(x,t) = C_0$
(non-unique equilibrium)



u(x)
u2
u1
x = 0
x = L
u(x)
x = 0
x = L

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,\;\; 0 \le t < \infty$$

General steady-state solution: $u(x,t) = C_0 + C_1 x$, where C_0, C_1 are constant.

Boundary conditions: $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = 1$.

$C_1 = 0,\; C_1 = 1 \implies$ **no equilibrium**

Homework

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q, \quad 0 \le x \le L, \;\; 0 \le t < \infty$$

Boundary conditions: $\frac{\partial u}{\partial x}(0,t) = u(0,t) - u_0$,
$\frac{\partial u}{\partial x}(L,t) = \alpha$.

Suppose $K_0 = \text{const}$ and $Q(x,t)/K_0 = x$,
$0 \le x \le L$, $t \ge 0$.

Problem. Find the steady-state solution of the boundary problem.

Solution

Let u be a steady-state solution of the heat equation. Then $u(x,t) = v(x)$, where $v : [0, L] \to \mathbb{R}$ satisfies the following ODE:

$$(K_0 v')' + Q = 0.$$

Since $K_0 = \text{const} > 0$, it follows that $v'' + Q/K_0 = 0$.

Hence $v''(x) + x = 0$ for $0 \le x \le L$.

$v''(x) = -x \implies v'(x) = -\frac{x^2}{2} + C_1 \implies$
$v(x) = -\frac{x^3}{6} + C_1 x + C_2$,

where C_1, C_2 are constants.

$$v'(x) = -x^2/2 + C_1,$$
$$v(x) = -x^3/6 + C_1 x + C_2, \quad 0 \le x \le L.$$

Boundary conditions are satisfied if $v'(0) = v(0) - u_0$ and $v'(L) = \alpha$.

That is, if $C_1 = C_2 - u_0$, $-L^2/2 + C_1 = \alpha$.

It follows that $C_1 = \alpha + L^2/2$, $C_2 = \alpha + L^2/2 + u_0$.

unique solution:

$$\begin{aligned} u(x,t) &= -x^3/6 + (\alpha + L^2/2)x + \alpha + L^2/2 + u_0 \\ &= -x^3/6 + (\alpha + L^2/2)(x+1) + u_0. \end{aligned}$$

New equation

$$\frac{\partial^2 u}{\partial w\,\partial z} = 0, \qquad u = u(w,z)$$

Domain: $a_1 \le w \le a_2$, $b_1 \le z \le b_2$.

(we allow intervals $[a_1, a_2]$ and $[b_1, b_2]$ to be infinite or semi-infinite)

$$\frac{\partial}{\partial w}\left(\frac{\partial u}{\partial z}\right) = 0, \qquad \frac{\partial u}{\partial z}(w,z) = \gamma(z)$$

$$u(w,z) = \int_{z_0}^{z} \gamma(\xi)\,d\xi + C(w)$$

$$\boxed{u(w,z) = B(z) + C(w)} \qquad \text{(general solution)}$$

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Change of independent variables:

$$w = x + ct, \qquad z = x - ct.$$

How does the equation look in new coordinates?

$$\frac{\partial}{\partial t} = \frac{\partial w}{\partial t}\frac{\partial}{\partial w} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z} = c\,\frac{\partial}{\partial w} - c\,\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial}{\partial w} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z} = \frac{\partial}{\partial w} + \frac{\partial}{\partial z}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial}{\partial w} - \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial w} - \frac{\partial}{\partial z}\right) u$$

$$= c^2 \left(\frac{\partial^2 u}{\partial w^2} - 2\frac{\partial^2 u}{\partial w\,\partial z} + \frac{\partial^2 u}{\partial z^2}\right).$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial w^2} + 2\frac{\partial^2 u}{\partial w\,\partial z} + \frac{\partial^2 u}{\partial z^2}.$$

$$\frac{\partial^2 u}{\partial t^2} - c^2\,\frac{\partial^2 u}{\partial x^2} = -4c^2\,\frac{\partial^2 u}{\partial w\,\partial z}.$$

Wave equation in new coordinates: $\dfrac{\partial^2 u}{\partial w\,\partial z} = 0.$

General solution: $\boxed{u(x,t) = B(x-ct) + C(x+ct)}$

(d'Alembert, 1747)