Math 412-501

Theory of Partial Differential Equations

Lecture 4: D'Alembert's solution (continued).

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad -\infty < t < \infty$$

Change of independent variables:

$$w = x + ct, \qquad z = x - ct.$$

Jacobian: $\begin{pmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial t} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix}$

How does the equation look in new coordinates?

$$\frac{\partial}{\partial t} = \frac{\partial w}{\partial t}\frac{\partial}{\partial w} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z} = c\frac{\partial}{\partial w} - c\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial}{\partial w} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z} = \frac{\partial}{\partial w} + \frac{\partial}{\partial z}$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2 \left(\frac{\partial}{\partial w} - \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial w} - \frac{\partial}{\partial z}\right) u \\
&= c^2 \left(\frac{\partial^2 u}{\partial w^2} - 2\frac{\partial^2 u}{\partial w\,\partial z} + \frac{\partial^2 u}{\partial z^2}\right).
\end{aligned}
$$

$$
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial w^2} + 2\frac{\partial^2 u}{\partial w\,\partial z} + \frac{\partial^2 u}{\partial z^2}.
$$

$$
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = -4c^2 \frac{\partial^2 u}{\partial w\,\partial z}.
$$

Wave equation in new coordinates: $\dfrac{\partial^2 u}{\partial w\,\partial z} = 0.$

$$\frac{\partial^2 u}{\partial w\, \partial z} = 0, \quad -\infty < w, z < \infty$$

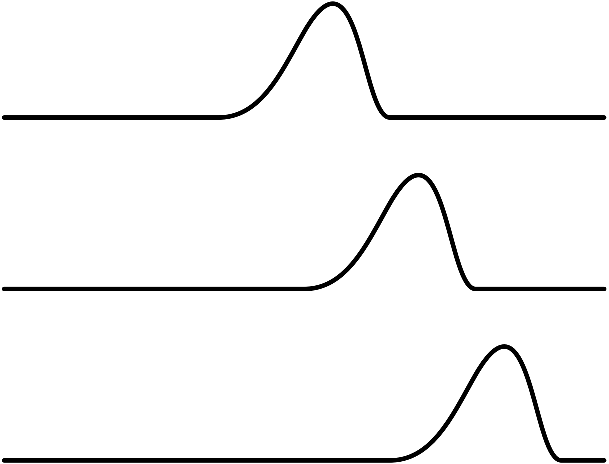
General solution: $\boxed{u(w,z) = B(z) + C(w)}$

where $B, C : \mathbb{R} \to \mathbb{R}$ are arbitrary (smooth) functions.

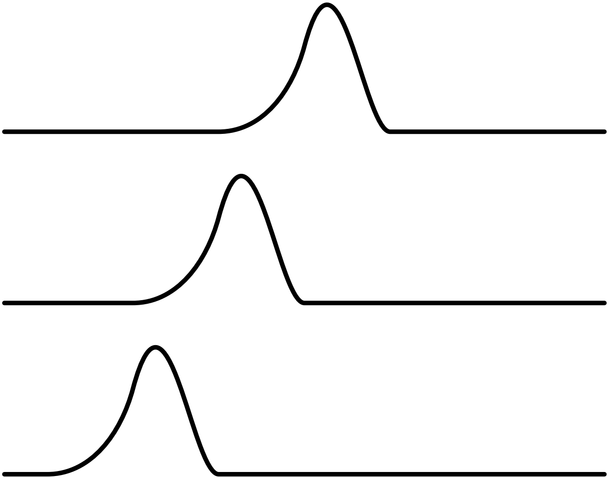
General solution of the 1D wave equation:

$$\boxed{u(x,t) = B(x - ct) + C(x + ct)}$$

(d'Alembert's solution)



$$u(x,t) = B(x - ct)$$
$$t_1 = 0,\ t_2 = 1,\ t_3 = 2$$



$$u(x,t) = C(x+ct)$$
$$t_1 = 0,\ t_2 = 1,\ t_3 = 2$$

Initial value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x, t < \infty,$$

$$u(x,0) = f(x), \;\; \frac{\partial u}{\partial t}(x,0) = g(x), \;\; -\infty < x < \infty.$$

General solution: $u(x,t) = B(x-ct) + C(x+ct)$.

Functions B and C are determined by the initial conditions:

$f(x) = B(x) + C(x), \qquad g(x) = -cB'(x) + cC'(x).$

$B + C = f, \; c(-B + C)' = g.$

$B + C = f$, $c(-B + C)' = g$.

$B + C = f$, $-B + C = G$, where $G' = g/c$ (G is determined up to adding a constant).

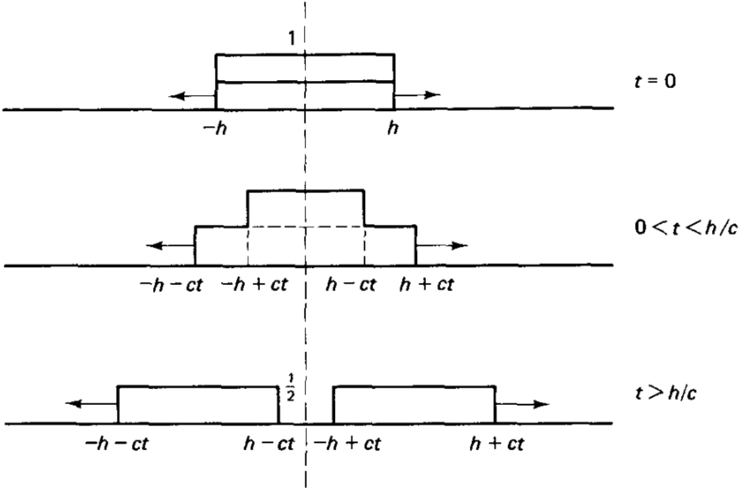
It follows that $B = \frac{1}{2}(f - G)$, $C = \frac{1}{2}(f + G)$.

$$\boxed{\begin{aligned} u(x,t) = \tfrac{1}{2}\Big(& f(x - ct) + f(x + ct) \\ & + G(x + ct) - G(x - ct)\Big) \end{aligned}}$$

(d'Alembert's formula)

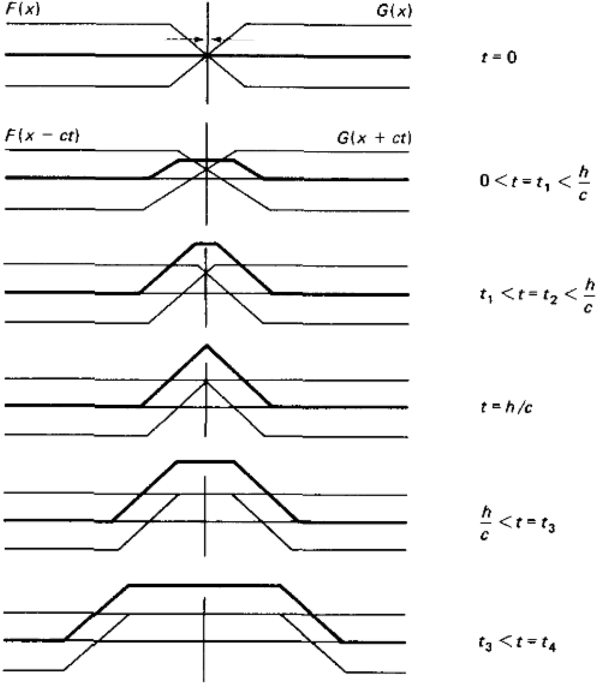
In this formula, G may be an arbitrary anti-derivative of g/c.

The solution is **unique**, but functions B and C are **not!**



$$f = \chi_{[-h,h]}$$
$$g = 0$$

$$f = 0$$
$$g = \chi_{[-h,h]}$$
$$G' = g/c$$
$$F = -G$$



$$u(x,t) = \frac{1}{2}\Big(f(x-ct) + f(x+ct) + G(x+ct) - G(x-ct)\Big).$$

Since $G' = g/c$, we have

$$G(x+ct) - G(x-ct) = \frac{1}{c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi.$$

$$\boxed{u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi}$$

(d'Alembert's formula)

Example

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x, t < \infty,$$

$$u(x,0) = \cos 2x, \;\; \frac{\partial u}{\partial t}(x,0) = \sin x, \;\; -\infty < x < \infty.$$

According to the (2nd) d'Alembert's formula, the unique solution is

$$u(x,t) = \tfrac{1}{2}\Big(f(x-ct) + f(x+ct) + G(x+ct) - G(x-ct) \Big),$$

where $f(x) = \cos 2x$, $x \in \mathbb{R}$, and G is an arbitrary function such that $G'(x) = \dfrac{\sin x}{c}$ for all $x \in \mathbb{R}$.

We can take $G(x) = -\dfrac{\cos x}{c}$. Then

$$u(x,t) = \tfrac{1}{2}\big(\cos 2(x - ct) + \cos 2(x + ct)\big) \\ + \tfrac{1}{2c}\big(-\cos(x + ct) + \cos(x - ct)\big).$$

After simplifying,

$$u(x,t) = \cos 2ct \cdot \cos 2x + \frac{1}{c}\sin ct \cdot \sin x.$$

Semi-infinite string

Initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad x \geq 0;$$

$$u(x,0) = f(x), \;\; \frac{\partial u}{\partial t}(x,0) = g(x), \;\; x \geq 0;$$

$$u(0,t) = 0 \;\; \text{(fixed end)}.$$

General solution: $u(x,t) = B(x - ct) + C(x + ct)$.

Initial conditions imply:

$f(x) = B(x) + C(x), \quad g(x) = -cB'(x) + cC'(x),$
$x \geq 0$.

$B + C = f$, $c(-B + C)' = g$.

$B + C = f$, $-B + C = G$, where $G' = g/c$
(G is determined up to adding a constant).

It follows that $B = \frac{1}{2}(f - G)$, $C = \frac{1}{2}(f + G)$.

However this yields $B(x)$ and $C(x)$ **only** for $x \geq 0$.

Boundary condition implies:

$B(-ct) + C(ct) = 0$ for all $t \in \mathbb{R}$.

That is, $B(-x) = -C(x)$ and $C(-x) = -B(x)$.

This yields $B(x)$ and $C(x)$ for $x < 0$.

Another approach

Initial value problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x, t < \infty,$$

$$u(x,0) = f(x), \;\; \frac{\partial u}{\partial t}(x,0) = g(x), \;\; -\infty < x < \infty.$$

Lemma Suppose that the functions f and g are **odd**, that is, $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all x.

Then the solution satisfies the fixed-end boundary condition at the origin: $u(0,t) = 0$ for all t.

Proof: By the (3rd) d'Alembert's formula,

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi.$$

Hence

$$u(0,t) = \frac{f(-ct) + f(ct)}{2} + \frac{1}{2c}\int_{-ct}^{ct} g(\xi)\, d\xi.$$

Since f is odd, we have $f(-ct) + f(ct) = 0$.
Since g is odd, we have

$$\int_{-ct}^{0} g(\xi)\, d\xi = -\int_{0}^{ct} g(\xi)\, d\xi$$

$$\Longrightarrow \int_{-ct}^{ct} g(\xi)\, d\xi = 0$$

Initial-boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad x \geq 0;$$

$$u(x,0) = f(x), \;\; \frac{\partial u}{\partial t}(x,0) = g(x), \;\; x \geq 0;$$

$$u(0,t) = 0 \;\; \text{(fixed end)}.$$

The problem can be solved as follows:

- extend f and g to the whole line so that they are odd;
- solve the initial value problem in the whole plane;
- restrict the solution to the half-plane $x \geq 0$.

Initial-boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad x \geq 0;$$

$$u(x,0) = f(x), \;\; \frac{\partial u}{\partial t}(x,0) = g(x), \;\; x \geq 0;$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \;\; \text{(free end)}.$$

The problem can be solved as follows:

- extend f and g to the whole line so that they are **even**: $f(-x) = f(x)$ and $g(-x) = g(x)$ for all x;
- solve the initial value problem in the whole plane;
- restrict the solution to the half-plane $x \geq 0$ (the boundary condition should hold).