Math 412-501

Theory of Partial Differential Equations

# Lecture 6: Separation of variables.

**How do we solve a linear homogeneous PDE?**

*Step 1:* Find some solutions.

*Step 2:* Form linear combinations of solutions obtained on Step 1.

*Step 3:* Show that every solution can be approximated by solutions obtained on Step 2.

Similarly, we solve a linear homogeneous PDE with linear homogeneous boundary conditions (boundary problem).

One way to complete Step 1: the method of **separation of variables**.

## Separation of variables

The method applies to certain linear PDEs. It is used to find some solutions.

**Basic idea:** to find a solution of the PDE (function of many variables) as a combination of several functions, each depending only on one variable.

For example, $u(x,t) = B(x) + C(t)$ or $u(x,t) = B(x)C(t)$.

The first example works perfectly for one equation: $\frac{\partial^2 u}{\partial t\, \partial x} = 0$.
The second example proved useful for **many** equations.

# Heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Suppose $u(x,t) = \phi(x)G(t)$. Then

$$\frac{\partial u}{\partial t} = \phi(x)\frac{dG}{dt}, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{d^2\phi}{dx^2}G(t).$$

Hence

$$\phi(x)\frac{dG}{dt} = k \frac{d^2\phi}{dx^2}G(t).$$

Divide both sides by $k \cdot \phi(x) \cdot G(t) = k \cdot u(x,t)$:

$$\frac{1}{kG} \cdot \frac{dG}{dt} = \frac{1}{\phi} \cdot \frac{d^2\phi}{dx^2}.$$

It follows that

$$\frac{1}{kG}\cdot\frac{dG}{dt}=\frac{1}{\phi}\cdot\frac{d^2\phi}{dx^2}=-\lambda=\text{const.}$$

$\lambda$ is called the **separation constant**. The variables have been separated:

$$\frac{d^2\phi}{dx^2}=-\lambda\phi,$$
$$\frac{dG}{dt}=-\lambda kG.$$

**Proposition** Suppose $\phi$ and $G$ are solutions of the above ODEs for the same value of $\lambda$. Then $u(x,t)=\phi(x)G(t)$ is a solution of the heat equation.

*Example.* $u(x,t)=e^{-kt}\sin x$.

$$\boxed{\frac{dG}{dt} = -\lambda k G}$$

General solution: $G(t) = C_0 e^{-\lambda k t}$, $C_0 = \text{const}$.

$$\boxed{\frac{d^2\phi}{dx^2} = -\lambda\phi}$$

Three cases: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

Case 1: $\lambda > 0$. Then $\lambda = \mu^2$, where $\mu > 0$.

$\phi(x) = C_1 \cos \mu x + C_2 \sin \mu x$, $C_1, C_2 = \text{const}$.

Case 2: $\lambda = 0$. $\phi(x) = C_1 + C_2 x$.

Case 3: $\lambda < 0$. Then $\lambda = -\mu^2$, where $\mu > 0$.

$\phi(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$.

**Theorem** For any $C_1, C_2 \in \mathbb{R}$ and $\mu > 0$, the functions

$$\begin{aligned} u_+(x,t) &= e^{-k\mu^2 t}(C_1 \cos \mu x + C_2 \sin \mu x), \\ u_0(x,t) &= C_1 + C_2 x, \\ u_-(x,t) &= e^{k\mu^2 t}(C_1 e^{\mu x} + C_2 e^{-\mu x}) \end{aligned}$$

are solutions of the heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}.$$

# Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Suppose $u(x,t) = \phi(x)G(t)$. Then

$$\frac{\partial^2 u}{\partial t^2} = \phi(x)\frac{d^2 G}{dt^2}, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{d^2\phi}{dx^2}G(t).$$

Hence

$$\phi(x)\frac{d^2 G}{dt^2} = c^2 \frac{d^2\phi}{dx^2}G(t).$$

Divide both sides by $c^2 \cdot \phi(x) \cdot G(t) = c^2 \cdot u(x,t)$:

$$\frac{1}{c^2 G} \cdot \frac{d^2 G}{dt^2} = \frac{1}{\phi} \cdot \frac{d^2\phi}{dx^2}.$$

It follows that

$$\frac{1}{c^2 G} \cdot \frac{d^2 G}{dt^2} = \frac{1}{\phi} \cdot \frac{d^2\phi}{dx^2} = -\lambda = \text{const.}$$

The variables have been separated:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi,$$

$$\frac{d^2 G}{dt^2} = -\lambda c^2 G.$$

**Proposition** Suppose $\phi$ and $G$ are solutions of the above ODEs for the same value of $\lambda$. Then $u(x,t) = \phi(x)G(t)$ is a solution of the wave equation.

*Example.* $u(x,t) = \cos ct \cdot \sin x$.

**Theorem** For any $C_1, C_2, D_1, D_2 \in \mathbb{R}$ and $\mu > 0$, the functions

$$\begin{aligned} u_+(x,t) &= (D_1 \cos c\mu t + D_2 \sin c\mu t) \\ &\quad \times (C_1 \cos \mu x + C_2 \sin \mu x), \\ u_0(x,t) &= (D_1 + D_2 t)(C_1 + C_2 x), \\ u_-(x,t) &= (D_1 e^{c\mu t} + D_2 e^{-c\mu t})(C_1 e^{\mu x} + C_2 e^{-\mu x}) \end{aligned}$$

are solutions of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

# Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

**Proposition** Suppose that

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \qquad \frac{d^2 h}{dy^2} = \lambda h,$$

where $\lambda = \text{const}$. Then $u(x,y) = \phi(x)h(y)$ is a solution of Laplace's equation.

*Proof:* $\frac{\partial^2 u}{\partial x^2} = \phi''(x)h(y) = -\lambda\phi(x)h(y)$,
$\frac{\partial^2 u}{\partial y^2} = \phi(x)h''(y) = \lambda\phi(x)h(y)$. Hence $\Delta u = 0$.

*Example.* $u(x,y) = e^y \sin x$.

# Boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$
$$u(0,t) = u(L,t) = 0.$$

We are looking for solutions $u(x,t) = \phi(x)G(t)$.

PDE holds if

$$\frac{d^2\phi}{dx^2} = -\lambda\phi,$$
$$\frac{dG}{dt} = -\lambda k G$$

for the same constant $\lambda$.

Boundary conditions hold if

$$\phi(0) = \phi(L) = 0.$$

Boundary value problem:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \qquad 0 \leq x \leq L,$$
$$\phi(0) = \phi(L) = 0.$$

There is an obvious solution: 0.
When is it **not unique?**

If for some value of $\lambda$ the boundary value problem has a nonzero solution $\phi$, then this $\lambda$ is called an **eigenvalue** and $\phi$ is called an **eigenfunction**.

The **eigenvalue problem** is to find all eigenvalues (and corresponding eigenfunctions).