Math 412-501

Theory of Partial Differential Equations

Lecture 8: Fourier's solution of the initial-boundary value problem (continued).

How do we solve the initial-boundary value problem?

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$

$$u(x,0) = f(x), \quad u(0,t) = u(L,t) = 0.$$

- Expand the function f into a series

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}.$$

- Write the solution:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \exp\Big(-\frac{n^2\pi^2}{L^2}kt\Big)\sin\frac{n\pi x}{L}.$$

(Fourier's solution)

How do we expand initial data into a series?

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \tfrac{n\pi x}{L}.$$

Assume that such an expansion exists and the series converges uniformly. Then

$$\int_0^L f(x)g(x)\,dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin \tfrac{n\pi x}{L}\, g(x)\,dx$$

for any continuous function $g : [0, L] \to \mathbb{R}$.
In particular, for $m = 1, 2, \ldots$ we have

$$\int_0^L f(x) \sin \tfrac{m\pi x}{L}\,dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin \tfrac{n\pi x}{L} \sin \tfrac{m\pi x}{L}\,dx.$$

It turns out that

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} \frac{1}{2}L & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Hence

$$\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx = B_m \cdot \frac{1}{2}L.$$

Therefore

$$B_m = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx.$$

How do we solve the initial-boundary value problem?

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$$u(x,0) = f(x), \quad u(0,t) = u(L,t) = 0.$$

- Expand the function f into a series

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L},$$

where

$$B_n = \frac{2}{L}\int_0^L f(\xi) \sin \frac{n\pi \xi}{L}\, d\xi.$$

- Write the solution:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left(-\frac{n^2\pi^2}{L^2}kt\right) \sin \frac{n\pi x}{L}.$$

Evaluation of an integral

$$\int_0^L \sin\tfrac{n\pi x}{L}\sin\tfrac{m\pi x}{L}\,dx = \frac{L}{\pi}\int_0^L \sin\tfrac{n\pi x}{L}\sin\tfrac{m\pi x}{L}\,d\left(\tfrac{\pi x}{L}\right)$$

$$= \frac{L}{\pi}\int_0^\pi \sin ny \cdot \sin my\,dy$$

$$= \frac{L}{2\pi}\int_0^\pi \Big(\cos\,(n-m)y - \cos\,(n+m)y\Big)\,dy.$$

Let $N \in \mathbb{Z}$. If $N \neq 0$ then

$$\int_0^\pi \cos Ny\,dy = \frac{\sin Ny}{N}\bigg|_0^\pi = 0.$$

If $N = 0$ then

$$\int_0^\pi \cos Ny\,dy = \int_0^\pi dy = \pi.$$

Example

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$
$$u(x,0) = 100, \quad u(0,t) = u(L,t) = 0.$$

Fourier's expansion:

$$100 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \qquad (0 < x < L),$$

where

$$B_n = \frac{2}{L}\int_0^L 100 \sin\frac{n\pi\xi}{L}\,d\xi = \frac{200}{\pi}\int_0^L \sin\frac{n\pi\xi}{L}\,d\left(\frac{\pi\xi}{L}\right)$$

$$= \frac{200}{\pi}\int_0^{\pi} \sin ny\,dy = \frac{200(1-\cos n\pi)}{n\pi}.$$

$B_n = 0$ if n is even, and $B_n = \frac{400}{n\pi}$ if n is odd.

n is odd $\implies n = 2m - 1$, m a positive integer.

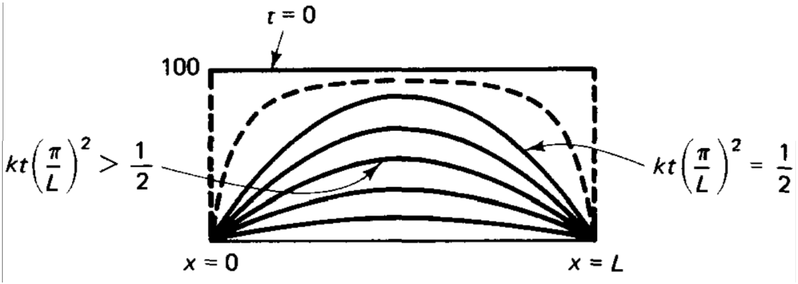
$$100 = \sum_{m=1}^{\infty} \frac{400}{(2m-1)\pi} \sin \frac{(2m-1)\pi x}{L}, \qquad 0 < x < L.$$

Fourier's solution:

$$u(x,t) = \sum_{m=1}^{\infty} \frac{400}{(2m-1)\pi} \exp\Big(-\frac{(2m-1)^2\pi^2}{L^2} kt\Big) \sin \frac{(2m-1)\pi x}{L}.$$

For a large t,

$$u(x,t) \approx \frac{400}{\pi} \exp\Big(-\frac{\pi^2}{L^2} kt\Big) \sin \frac{\pi x}{L}.$$


$t = 0$
100
$kt\left(\frac{\pi}{L}\right)^2 > \frac{1}{2}$
$kt\left(\frac{\pi}{L}\right)^2 = \frac{1}{2}$
$x = 0$
$x = L$

More boundary conditions for the heat equation

Initial-boundary value problem:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$

$$u(x,0) = f(t), \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0.$$

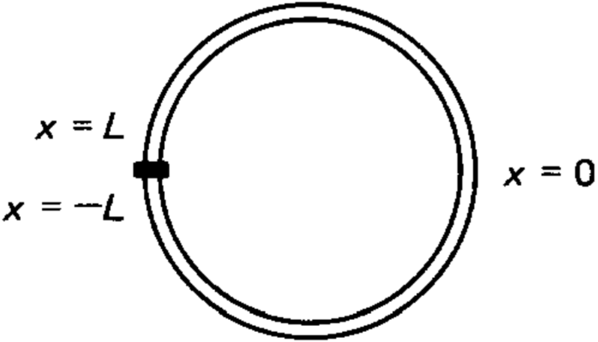
(insulated ends)

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad -L \le x \le L,$$

$$u(x,0) = f(t),$$
$$u(-L,t) = u(L,t), \quad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t).$$

(periodic boundary conditions)

Heat conduction in a thin circular ring



Separation of variables: $u(x,t) = \phi(x)G(t)$.

PDE holds if for some $\lambda = \text{const}$,

$$\frac{d^2\phi}{dx^2} = -\lambda\phi,$$

$$\frac{dG}{dt} = -\lambda k G \implies G(t) = C_0 \exp(-\lambda k t).$$

Boundary conditions $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$ hold if

$$\phi'(0) = \phi'(L) = 0.$$

Boundary conditions $u(-L,t) = u(L,t)$, $\frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t) = 0$ hold if

$$\phi(-L) = \phi(L), \quad \phi'(-L) = \phi'(L).$$

Eigenvalue problem (insulated ends):

$$\phi'' = -\lambda\phi, \qquad \phi'(0) = \phi'(L) = 0.$$

Three cases: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

Case 1: $\lambda > 0$. $\quad \phi(x) = C_1 \cos\mu x + C_2 \sin\mu x$,
where $\lambda = \mu^2$, $\mu > 0$.

$\phi'(0) = \phi'(L) = 0 \implies C_2 = 0, \; -C_1\mu \sin\mu L = 0.$
A nonzero solution exists if $\mu L = n\pi$, $n \in \mathbb{Z}$.

So $\lambda_n = (\frac{n\pi}{L})^2$, $n = 1, 2, \dots$ are eigenvalues and $\phi_n(x) = \cos\frac{n\pi x}{L}$ are corresponding eigenfunctions.

The only other eigenvalue is $\lambda_0 = 0$, with the eigenfunction $\phi_0 = 1$.

Eigenvalue problem (circular ring):

$$\phi'' = -\lambda\phi, \quad \phi(-L) = \phi(L), \;\; \phi'(-L) = \phi'(L).$$

Case 1: $\lambda > 0$. $\quad \phi(x) = C_1 \cos\mu x + C_2 \sin\mu x$,
where $\lambda = \mu^2$, $\mu > 0$.

$$\phi(-L) = \phi(L) \implies C_2 \sin\mu L = 0.$$
$$\phi'(-L) = \phi'(L) \implies -C_1\mu \sin\mu L = 0.$$

A nonzero solution exists if $\mu L = n\pi$, $n \in \mathbb{Z}$.

So $\lambda_n = (\frac{n\pi}{L})^2$, $n = 1, 2, \dots$ are **multiple** eigenvalues while $\phi_n(x) = \cos\frac{n\pi x}{L}$ and $\psi_n(x) = \sin\frac{n\pi x}{L}$ are corresponding eigenfunctions.

The only other eigenvalue is $\lambda_0 = 0$, with the eigenfunction $\phi_0 = 1$.

Fourier's solution (insulated ends)

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0.$$

- Expand the function f into a series

$$f(x) = A_0 + \sum\nolimits_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}.$$

- Write the solution:

$$u(x,t) = A_0 + \sum\nolimits_{n=1}^{\infty} A_n \exp\Big(-\frac{n^2\pi^2}{L^2}kt\Big)\cos\frac{n\pi x}{L}.$$

Fourier's solution (circular ring)

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad -L \le x \le L,$$

$$u(x,0) = f(t),$$
$$u(-L,t) = u(L,t), \quad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t).$$

- Expand the function f into a series

$$f(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right).$$

- Write the solution:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} \exp\left(-\tfrac{n^2\pi^2}{L^2}kt\right)\left(A_n \cos\tfrac{n\pi x}{L} + B_n \sin\tfrac{n\pi x}{L}\right).$$