Math 412-501

Theory of Partial Differential Equations

Lecture 2-10: Sturm-Liouville eigenvalue problems (continued). Hilbert space.

Regular Sturm-Liouville eigenvalue problem:

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0 \qquad (a < x < b),$$

$$\beta_1\phi(a) + \beta_2\phi'(a) = 0,$$

$$\beta_3\phi(b) + \beta_4\phi'(b) = 0.$$

Here $\beta_i \in \mathbb{R}$, $|\beta_1| + |\beta_2| \neq 0$, $|\beta_3| + |\beta_4| \neq 0$.
Functions p, q, σ are continuous on $[a, b]$,
$p > 0$ and $\sigma > 0$ on $[a, b]$.

6 properties of a regular Sturm-Liouville problem

- Eigenvalues are real.
- Eigenvalues form an increasing sequence.
- n-th eigenfunction has $n - 1$ zeros in (a, b).
- Eigenfunctions are orthogonal with weight σ.
- Eigenfunctions and eigenvalues are related through the Rayleigh quotient.
- Piecewise smooth functions can be expanded into generalized Fourier series of eigenfunctions.

Hilbert space

Hilbert space is an infinite-dimensional analog of Euclidean space. One realization is

$$L_2[a, b] = \{f : \int_a^b |f(x)|^2\, dx < \infty\}.$$

Inner product of functions:

$$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx.$$

If f and g take complex values, then

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\, dx$$

so that

$$\langle f, f \rangle = \int_a^b |f(x)|^2\, dx \geq 0.$$

Norm of a function: $\|f\| = \sqrt{\langle f, f \rangle}$.

Cauchy-Schwarz inequality: $|\langle f, g \rangle| \leq \|f\| \cdot \|g\|$.

If f, g are real-valued, then $\langle f, g \rangle = \|f\| \cdot \|g\| \cos \theta$, where θ is called the **angle** between f and g.

Convergence: we say that $f_n \to f$ **in the mean** if $\|f - f_n\| \to 0$ as $n \to \infty$.

Lemma If $f_n \to f$ in the mean then $\langle f_n, g \rangle \to \langle f, g \rangle$ for any $g \in L_2[a, b]$.

Proof:
$|\langle f, g \rangle - \langle f_n, g \rangle| = |\langle f - f_n, g \rangle| \leq \|f - f_n\| \cdot \|g\|$.

Functions $f, g \in L_2[a, b]$ are called **orthogonal** if $\langle f, g \rangle = 0$.

Alternative inner product:

$$\langle f, g \rangle_w = \int_a^b f(x)g(x)w(x)\,dx,$$

where $w > 0$ is the **weight** function.

Functions f and g are called **orthogonal with weight** w if $\langle f, g \rangle_w = 0$.

Alternative inner product means an alternative model of the Hilbert space:

$$L_2([a, b], w\,dx) = \{f : \int_a^b |f(x)|^2 w(x)\,dx < \infty\}.$$

A set $f_1, f_2, \ldots$ of pairwise orthogonal nonzero functions is called **complete** if it is maximal, i.e., there is no nonzero function g such that $\langle g, f_n \rangle = 0$, $n = 1, 2, \ldots$.

A complete set forms a **basis** of the Hilbert space, that is, each function $g \in L_2[a, b]$ can be expanded into a series

$$g = \sum_{n=1}^{\infty} c_n f_n$$

that converges in the mean.

Then

$$\langle g, h \rangle = \sum_{n=1}^{\infty} c_n \langle f_n, h \rangle$$

for any $h \in L_2[a, b]$.

In particular,

$$\langle g, f_m \rangle = \sum_{n=1}^{\infty} c_n \langle f_n, f_m \rangle = c_m \langle f_m, f_m \rangle.$$

$\Longrightarrow$ the expansion is unique: $c_m = \dfrac{\langle g, f_m \rangle}{\langle f_m, f_m \rangle}$.

Also,

$$\langle g, g \rangle = \sum_{n=1}^{\infty} c_n \langle f_n, g \rangle = \sum_{n=1}^{\infty} |c_n|^2 \langle f_n, f_n \rangle.$$

$$\boxed{\langle g, g \rangle = \sum_{n=1}^{\infty} \frac{|\langle g, f_n \rangle|^2}{\langle f_n, f_n \rangle}}$$

(Parseval's equality)

Suppose that $f_1, f_2, \ldots$ is an **orthonormal** basis, i.e., $\|f_n\| = 1$. Then

$$g = \sum_{n=1}^{\infty} c_n f_n, \quad \text{where} \quad c_n = \langle g, f_n \rangle.$$

Parseval's equality becomes $\|g\|^2 = \sum_{n=1}^{\infty} |c_n|^2$.

If $h = \sum_{n=1}^{\infty} d_n f_n$, then

$$\langle g, h \rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_n \overline{d_m} \langle f_n, f_m \rangle = \sum_{n=1}^{\infty} c_n \overline{d_n}.$$

Which sequences $c_1, c_2, \ldots$ are allowed as coefficients of an expansion?

Theorem For any sequence $c_1, c_2, \dots$ such that $\sum_{n=1}^{\infty} |c_n|^2 < \infty$, the series

$$\sum_{n=1}^{\infty} c_n f_n$$

converges in the mean to some function $g \in L_2[a, b]$.

This gives rise to another model of the Hilbert space: $\ell_2 = \{(c_1, c_2, \dots) : \sum_{n=1}^{\infty} |c_n|^2 < \infty\}$.

Given $\mathbf{c} = (c_1, c_2, \dots)$, $\mathbf{d} = (d_1, d_2, \dots) \in \ell_2$, the inner product is

$$\langle \mathbf{c}, \mathbf{d} \rangle = \sum_{n=1}^{\infty} c_n \overline{d_n}.$$

Suppose $f_1, f_2, \dots$ is a set of pairwise orthogonal nonzero functions in $L_2[a,b]$ that is not complete.

For any function $g \in L_2[a,b]$, we can still compose a series $\sum_{n=1}^{\infty} c_n f_n$, where $c_n = \dfrac{\langle g, f_n \rangle}{\langle f_n, f_n \rangle}$.

This series converges in the mean to some function $g_0 \in L_2[a,b]$. In general, $g \neq g_0$ but $g - g_0$ is orthogonal to $f_1, f_2, \dots$.

Then $g = \sum_{n=1}^{\infty} c_n f_n + (g - g_0)$ implies
$\|g\|^2 = \sum_{n=1}^{\infty} \|c_n f_n\|^2 + \|g - g_0\|^2 \geq \sum_{n=1}^{\infty} \|c_n f_n\|^2$.

Bessel's inequality: $$\boxed{\langle g, g \rangle \geq \sum_{n=1}^{\infty} \frac{|\langle g, f_n \rangle|^2}{\langle f_n, f_n \rangle}}$$

$\mathcal{L}$: linear operator in the Hilbert space $L_2[a, b]$.

In general, $\mathcal{L}$ is not defined on the whole space but on a linear subspace $\mathcal{H} \subset L_2[a, b]$ which is dense.

Example. $\mathcal{L}(f) = (pf')' + qf$.

$\mathcal{L}$ is called **self-adjoint** (or **symmetric**) if

$$\langle \mathcal{L}(f), g \rangle = \langle f, \mathcal{L}(g) \rangle \quad \text{for all } \ f, g \in \mathcal{H}.$$

If $\mathcal{L}(f) = \lambda f$ for some $\lambda \in \mathbb{C}$ and nonzero $f \in \mathcal{H}$, then λ is an **eigenvalue** and f is an **eigenfunction**.

If the operator $\mathcal{L}$ is self-adjoint, then

- all eigenvalues are real;
- eigenfunctions belonging to different eigenvalues are orthogonal.

Regular Sturm-Liouville equation:

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0 \qquad (a < x < b).$$

Consider a linear differential operator

$$\mathcal{L}(f) = (pf')' + qf.$$

Now the equation can be rewritten as

$$\mathcal{L}(\phi) + \lambda\sigma\phi = 0.$$

Green's formula:

$$\int_a^b \Big(g\mathcal{L}(f) - f\mathcal{L}(g)\Big)\,dx = p(gf' - fg')\,\Big|_a^b$$

If f and g satisfy the same regular boundary conditions, then

$$\int_a^b \Big(g\mathcal{L}(f) - f\mathcal{L}(g) \Big)\, dx = 0.$$

That is, $\mathcal{L}$ is self-adjoint on the set of functions satisfying particular boundary conditions.

$$\mathcal{L}(\phi) + \lambda\sigma\phi = 0 \implies -\sigma^{-1}\mathcal{L}(\phi) = \lambda\phi$$

So eigenvalues/eigenfunctions of the Sturm-Liouville problem are not those of operator $\mathcal{L}$ but those of operator $\mathcal{M} = -\sigma^{-1}\mathcal{L}$.

The operator $\mathcal{M}$ is self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_\sigma$.

Eigenvalue problem:

$$\phi'' + \lambda\phi = 0, \quad \phi(0) = \phi(L) = 0.$$

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2$, $n = 1, 2, \dots$

Eigenfunctions: $\phi_n(x) = \sin \frac{n\pi x}{L}$.

Since this is a regular Sturm-Liouville problem, eigenfunctions form a complete orthogonal set (a basis) in the Hilbert space $L_2[0, L]$.

Any function $f \in L_2[0, L]$ is expanded into a series

$$f = \sum_{n=1}^{\infty} c_n \phi_n$$

that converges in the mean.

Coefficients:

$$c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

So the Fourier series always converges in the mean.

Parseval's equality:

$$\langle f, f \rangle = \sum_{n=1}^{\infty} \frac{|\langle f, \phi_n \rangle|^2}{\langle \phi_n, \phi_n \rangle} = \sum_{n=1}^{\infty} |c_n|^2 \langle \phi_n, \phi_n \rangle.$$

$$\boxed{\frac{2}{L} \int_0^L |f(x)|^2 \, dx = \sum_{n=1}^{\infty} |c_n|^2}$$

(Parseval's equality for Fourier sine series)

Example. $f(x) = 2x, \;\; 0 \le x \le \pi.$

$$f(x) \sim \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{n} \sin nx$$

Parseval's equality:

$$\frac{2}{\pi} \int_0^{\pi} (2x)^2 \, dx = \sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} \frac{16}{n^2}.$$

$$\frac{2}{\pi} \cdot \frac{4\pi^3}{3} = \sum_{n=1}^{\infty} \frac{16}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Simplicity of eigenvalues

Regular Sturm-Liouville equation:

$$(p\phi')' + q\phi + \lambda\sigma\phi = 0 \qquad (a < x < b).$$

Initial value problem $\phi(a) = C_0$, $\phi'(a) = C_1$ always has a unique solution.

Suppose ϕ and ψ are eigenfunctions of a regular problem corresponding to the same eigenvalue λ.

Then $\beta_1\phi(a) + \beta_2\phi'(a) = \beta_1\psi(a) + \beta_2\psi'(a) = 0$, where $\beta_1, \beta_2 \in \mathbb{R}$, $|\beta_1| + |\beta_2| \neq 0$.

It follows that $(\phi(a), \phi'(a)) = c(\psi(a), \psi'(a))$, $c \in \mathbb{R}$.

Now ϕ and $c\psi$ are solutions to the same initial value problem. Hence $\phi = c\psi$.