Math 412-501

Theory of Partial Differential Equations

Lecture 2-11: Review for Exam 2.

Heat conduction in a rectangle

Initial-boundary value problem:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (0 < x < L,\ 0 < y < H),$$

$$u(x, y, 0) = f(x, y) \quad (0 < x < L,\ 0 < y < H),$$

$$\frac{\partial u}{\partial x}(0, y, t) = \frac{\partial u}{\partial x}u(L, y, t) = 0 \quad (0 < y < H),$$

$$\frac{\partial u}{\partial y}(x, 0, t) = \frac{\partial u}{\partial y}u(x, H, t) = 0 \quad (0 < x < L).$$

We search for the solution $u(x, y, t)$ as a superposition of solutions with separated variables that satisfy the boundary conditions.

Separation of variables: $u(x,y,t) = \phi(x)h(y)G(t)$.
Substitute this into the heat equation:

$$\phi(x)h(y)\frac{dG}{dt} = k\left(\frac{d^2\phi}{dx^2}h(y)G(t) + \phi(x)\frac{d^2h}{dy^2}G(t)\right).$$

Divide both sides by $k \cdot \phi(x)h(y)G(t) = k \cdot u(x,y,t)$:

$$\frac{1}{kG} \cdot \frac{dG}{dt} = \frac{1}{\phi} \cdot \frac{d^2\phi}{dx^2} + \frac{1}{h} \cdot \frac{d^2h}{dy^2}.$$

It follows that $\frac{1}{\phi} \cdot \frac{d^2\phi}{dx^2} = -\lambda$, $\frac{1}{h} \cdot \frac{d^2h}{dy^2} = -\mu$,
$\frac{1}{kG} \cdot \frac{dG}{dt} = -\lambda - \mu$, where λ and μ are **separation constants**.

The variables have been separated:

$$\frac{dG}{dt} = -(\lambda + \mu)kG,$$

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \qquad \frac{d^2h}{dy^2} = -\mu h.$$

Proposition Suppose G, ϕ, and h are solutions of the above ODEs for the same values of λ and μ. Then $u(x,y,t) = \phi(x)h(y)G(t)$ is a solution of the heat equation.

Boundary conditions $\frac{\partial u}{\partial x}(0,y,t) = \frac{\partial u}{\partial x}(L,y,t) = 0$ hold if $\phi'(0) = \phi'(L) = 0$.

Boundary conditions $\frac{\partial u}{\partial y}(x,0,t) = \frac{\partial u}{\partial y}(x,H,t) = 0$ hold if $h'(0) = h'(H) = 0$.

1st eigenvalue problem: $\phi'' = -\lambda\phi,\;\; \phi'(0) = \phi'(L) = 0.$

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2,\; n = 0, 1, 2, \ldots$

Eigenfunctions: $\phi_0 = 1,\; \phi_n(x) = \cos\frac{n\pi x}{L},\; n \geq 1.$

2nd eigenvalue problem: $h'' = -\mu h,\;\; h'(0) = h'(H) = 0.$

Eigenvalues: $\mu_m = (\frac{m\pi}{H})^2,\; m = 0, 1, 2, \ldots$

Eigenfunctions: $h_0 = 1,\; h_m(y) = \cos\frac{m\pi y}{H},\; m \geq 1.$

Dependence on t:

$$G'(t) = -(\lambda + \mu)kG(t) \implies G(t) = C_0 e^{-(\lambda+\mu)kt}$$

Solution of the boundary value problem:

$$u(x, y, t) = e^{-(\lambda_n+\mu_m)kt}\phi_n(x)h_m(y)$$

$$= \exp\big(-((\tfrac{n\pi}{L})^2 + (\tfrac{m\pi}{H})^2)kt\big) \cdot \cos\frac{n\pi x}{L} \cdot \cos\frac{m\pi y}{H}.$$

We are looking for the solution of the initial-boundary value problem as a superposition of solutions with separated variables.

$$u(x,y,t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{n,m} e^{-(\lambda_n+\mu_m)kt} \phi_n(x) h_m(y)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{n,m} \exp\Big(-((\tfrac{n\pi}{L})^2 + (\tfrac{m\pi}{H})^2)kt\Big) \cdot \cos\frac{n\pi x}{L} \cdot \cos\frac{m\pi y}{H}$$

How do we find coefficients $C_{n,m}$?
From the initial condition $u(x,y,0) = f(x,y)$.

$$f(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{n,m} \cos\frac{n\pi x}{L} \cos\frac{m\pi y}{H}$$

(double Fourier cosine series)

How do we solve the heat conduction problem?

$$\frac{\partial u}{\partial t} = k\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) \quad (0 < x < L,\, 0 < y < H),$$

$$u(x, y, 0) = f(x, y) \quad (0 < x < L,\, 0 < y < H),$$

$$\tfrac{\partial u}{\partial x}(0, y, t) = \tfrac{\partial u}{\partial x}(L, y, t) = 0 \quad (0 < y < H),$$

$$\tfrac{\partial u}{\partial y}(x, 0, t) = \tfrac{\partial u}{\partial y}(x, H, t) = 0 \quad (0 < x < L).$$

- Expand f into the double Fourier cosine series:

$$f(x, y) = \sum\nolimits_{n=0}^{\infty} \sum\nolimits_{m=0}^{\infty} C_{n,m} \cos \tfrac{n\pi x}{L} \cos \tfrac{m\pi y}{H}.$$

- Write the solution: $u(x, y, t) =$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{n,m} \exp\Big(-((\tfrac{n\pi}{L})^2 + (\tfrac{m\pi}{H})^2)kt\Big) \cos \tfrac{n\pi x}{L} \cos \tfrac{m\pi y}{H}.$$

How do we expand f into the double Fourier cosine series?

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{n,m} \cos \frac{n\pi x}{L} \cos \frac{m\pi y}{H}.$$

If $f(x, y)$ is smooth then the expansion **exists**.

How do we determine coefficients?

Multiply both sides by $\phi_N(x) h_M(y)$ and integrate over the rectangle.

$$\int_0^L \int_0^H f(x,y)\phi_N(x)h_M(y)\,dx\,dy$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{n,m} \int_0^L \int_0^H \phi_N(x)h_M(y)\phi_n(x)h_m(y)\,dx\,dy$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{n,m} \int_0^L \phi_N(x)\phi_n(x)\,dx \int_0^H h_M(y)h_m(y)\,dy.$$

We have the orthogonality relations:

$$\int_0^L \phi_N(x)\phi_n(x)\,dx = 0, \qquad n \neq N,$$

$$\int_0^H h_M(y)h_m(y)\,dy = 0, \qquad m \neq M.$$

Hence

$$\int_0^L \int_0^H f(x,y)\phi_N(x)h_M(y)\,dx\,dy$$

$$= C_{N,M}\int_0^L \phi_N^2(x)\,dx \int_0^H h_M^2(y)\,dy.$$

It remains to recall that

$$\int_0^L \phi_0^2(x)\,dx = L, \qquad \int_0^L \phi_N^2(x)\,dx = \frac{L}{2}, \quad N \geq 1,$$

$$\int_0^H h_0^2(x)\,dx = H, \qquad \int_0^H h_M^2(x)\,dx = \frac{H}{2}, \quad M \geq 1.$$

How do we expand $f(x,y)$ into the double series?

$$f(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{n,m} \cos\frac{n\pi x}{L} \cos\frac{m\pi y}{H},$$

where

$$C_{0,0} = \frac{1}{LH} \int_0^L \int_0^H f(x,y)\,dx\,dy,$$

$$C_{n,0} = \frac{2}{LH} \int_0^L \int_0^H f(x,y) \cos\frac{n\pi x}{L}\,dx\,dy, \quad n \geq 1,$$

$$C_{0,m} = \frac{2}{LH} \int_0^L \int_0^H f(x,y) \cos\frac{m\pi y}{H}\,dx\,dy, \quad m \geq 1,$$

$$C_{n,m} = \frac{4}{LH} \int_0^L \int_0^H f(x,y) \cos\frac{n\pi x}{L} \cos\frac{m\pi y}{H}\,dx\,dy.$$

Laplace's equation in a semicircle

Solve Laplace's equation inside a semicircle of radius a ($0 < r < a$, $0 < \theta < \pi$) subject to the boundary conditions: $u = 0$ on the diameter and $u(a, \theta) = g(\theta)$.

Laplace's equation in polar coordinates (r, θ):

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

$u = 0$ on the diameter $\implies$

$$u(r, 0) = u(r, \pi) = 0 \qquad (0 < r < a),$$

$$u(0, \theta) = 0 \qquad (0 < \theta < \pi).$$

Separation of variables: $u(r,\theta) = h(r)\phi(\theta)$.
Substitute this into Laplace's equation:

$$\frac{d^2h}{dr^2}\,\phi(\theta) + \frac{1}{r}\,\frac{dh}{dr}\,\phi(\theta) + \frac{1}{r^2}\,h(r)\,\frac{d^2\phi}{d\theta^2} = 0.$$

Divide both sides by $r^{-2}h(r)\phi(\theta) = r^{-2}u(r,\theta)$:

$$\frac{1}{h}\cdot\left(r^2\,\frac{d^2h}{dr^2} + r\,\frac{dh}{dr}\right) = -\frac{1}{\phi}\cdot\frac{d^2\phi}{d\theta^2}.$$

It follows that

$$\frac{1}{h}\cdot\left(r^2\,\frac{d^2h}{dr^2} + r\,\frac{dh}{dr}\right) = -\frac{1}{\phi}\cdot\frac{d^2\phi}{d\theta^2} = \lambda = \text{const}.$$

The variables have been separated:

$$r^2 \frac{d^2h}{dr^2} + r\frac{dh}{dr} = \lambda h, \qquad \frac{d^2\phi}{d\theta^2} = -\lambda\phi.$$

Proposition Suppose h and ϕ are solutions of the above ODEs for the same value of λ. Then $u(r,\theta) = h(r)\phi(\theta)$ is a solution of Laplace's equation.

Boundary conditions $u(r,0) = u(r,\pi) = 0$ hold if

$$\phi(0) = \phi(\pi) = 0.$$

Boundary condition $u(0,\theta) = 0$ holds if

$$h(0) = 0.$$

Euler's (or equidimensional) equation

$$r^2 \frac{d^2 h}{dr^2} + r \frac{dh}{dr} - \lambda h = 0 \quad (r > 0)$$

$$\lambda > 0 \implies h(r) = C_1 r^p + C_2 r^{-p} \quad (\lambda = p^2, \ p > 0)$$

$$\lambda = 0 \implies h(r) = C_1 + C_2 \log r$$

Eigenvalue problem: $\phi'' = -\lambda\phi$, $\phi(0) = \phi(\pi) = 0$.

Eigenvalues: $\lambda_n = n^2$, $n = 1, 2, \ldots$

Eigenfunctions: $\phi_n(\theta) = \sin n\theta$.

Dependence on r:

$$r^2 h'' + rh' = \lambda h, \qquad h(0) = 0.$$

$$\Longrightarrow \; h(r) = C_0 r^p \;\; (p = \sqrt{\lambda})$$

Solution of Laplace's equation:

$$u(r, \theta) = r^n \sin n\theta, \qquad n = 1, 2, \ldots$$

A superposition of the solutions with separated variables is a series

$$u(r,\theta) = \sum_{n=1}^{\infty} c_n r^n \sin n\theta,$$

where $c_1, c_2, \dots$ are constants.

Substituting the series into the boundary condition $u(a,\theta) = g(\theta)$, we get

$$g(\theta) = \sum_{n=1}^{\infty} c_n a^n \sin n\theta.$$

Hence $c_n = b_n a^{-n}, \quad n = 1, 2, \dots,$ where

$$g(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta.$$

Solution.

$$u(r,\theta) = \sum_{n=1}^{\infty} b_n \left(\frac{r}{a}\right)^n \sin n\theta,$$

where

$$\sum_{n=1}^{\infty} b_n \sin n\theta$$

is the Fourier sine series of the function $g(\theta)$ on $[0,\pi]$, that is,

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(\theta) \sin n\theta \, d\theta, \quad n = 1, 2, \ldots$$