Math 412-501

Theory of Partial Differential Equations

Lecture 2-2: Higher-dimensional wave equation. Complex-valued functions and Laplace's equation.

One-dimensional heat equation

Describes heat conduction in a rod:

$$\boxed{c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q}$$

$K_0 = K_0(x)$, $c = c(x)$, $\rho = \rho(x)$, $Q = Q(x, t)$.

Assuming K_0, c, ρ are constant (uniform rod) and $Q = 0$ (no heat sources), we obtain

$$\boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}}$$

where $k = K_0(c\rho)^{-1}$.

Higher-dimensional heat equation

$$\boxed{c\rho \frac{\partial u}{\partial t} = \nabla \cdot (K_0 \nabla u) + Q}$$

Assuming $K_0 = \text{const}$, we have

$$c\rho \frac{\partial u}{\partial t} = K_0 \nabla^2 u + Q,$$

where $\nabla^2 u = \nabla \cdot (\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Assuming $K_0, c, \rho = \text{const}$ (uniform medium) and $Q = 0$ (no heat sources), we obtain

$$\boxed{\frac{\partial u}{\partial t} = k \nabla^2 u,}$$

where $k = K_0 (c\rho)^{-1}$ is called the *thermal diffusivity*.

Notation

Each function $f : \mathbb{R}^3 \to \mathbb{R}$ is assigned the gradient (a vector field) and the Laplacian (a function). Each vector field $\vec{\phi} : \mathbb{R}^3 \to \mathbb{R}^3$ is assigned the divergence (a function).

"physical" notation: $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

gradient: $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$

divergence: $\nabla \cdot \vec{\phi} = \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_z}{\partial z}$

Laplacian: $\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

"mathematical" notation:

gradient: $\operatorname{grad} f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$

divergence: $\operatorname{div} \vec{\phi} = \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_z}{\partial z}$

Laplacian: $\Delta f = \operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

More notation

Each vector field $\vec{\phi} : \mathbb{R}^3 \to \mathbb{R}^3$ is assigned the **curl** (another vector field).

If $\vec{\phi} = (\phi_x, \phi_y, \phi_z)$ then

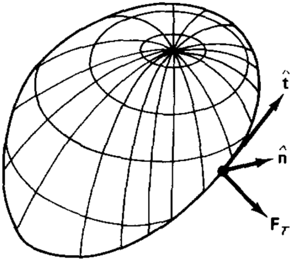
$$\operatorname{curl} \vec{\phi} = \left(\frac{\partial \phi_z}{\partial y} - \frac{\partial \phi_y}{\partial z}, \; \frac{\partial \phi_x}{\partial z} - \frac{\partial \phi_z}{\partial x}, \; \frac{\partial \phi_y}{\partial x} - \frac{\partial \phi_x}{\partial y} \right).$$

"physical" notation:

$$\nabla \times \phi = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi_x & \phi_y & \phi_z \end{vmatrix},$$

where $\mathbf{x} = (1,0,0)$, $\mathbf{y} = (0,1,0)$, $\mathbf{z} = (0,0,1)$.

Vibration of a stretched membrane

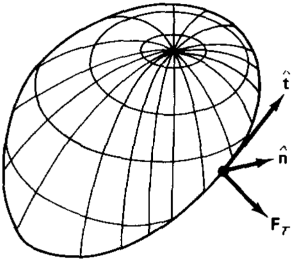


$u(x, y, t)$ = vertical displacement

Newton's law: mass × acceleration = force

$\rho(x, y)$ = mass density

$T(x, y, t)$ = magnitude of tensile force
$Q(x, y, t)$ = other (vertical) forces on a unit mass



$\vec{F}$ = tensile force

$\vec{F} = T(x, y, t)\,\mathbf{t} \times \mathbf{n}$

vertical component $= \vec{F} \cdot \mathbf{z}$

mass $\times$ acceleration:

$$\iint_D \rho(x,y)\frac{\partial^2 u}{\partial t^2}\,dx\,dy = \iint_D \rho\,\frac{\partial^2 u}{\partial t^2}\,dA$$

$$\text{tensile force} = \oint_{\partial D} \vec{F}\cdot\mathbf{z}\,ds = \oint_{\partial D} T(\mathbf{t}\times\mathbf{n})\cdot\mathbf{z}\,ds$$

$$\text{other forces} = \iint_D \rho Q\,dA$$

Newton's law:

$$\iint_D \rho\,\frac{\partial^2 u}{\partial t^2}\,dA = \oint_{\partial D} T(\mathbf{t}\times\mathbf{n})\cdot\mathbf{z}\,ds + \iint_D \rho Q\,dA$$

Since $(\mathbf{t} \times \mathbf{n}) \cdot \mathbf{z} = (\mathbf{n} \times \mathbf{z}) \cdot \mathbf{t}$,

$$\iint_D \rho \frac{\partial^2 u}{\partial t^2}\, dA = \oint_{\partial D} T(\mathbf{n} \times \mathbf{z}) \cdot \mathbf{t}\, ds + \iint_D \rho Q\, dA.$$

For any vector field $\vec{B}$,

$$\boxed{\iint_D (\nabla \times \vec{B}) \cdot \mathbf{n}\, dA = \oint_{\partial D} \vec{B} \cdot \mathbf{t}\, ds}$$

(Stokes' theorem)

$$\iint_D \rho \frac{\partial^2 u}{\partial t^2}\, dA = \iint_D \big(\nabla \times T(\mathbf{n} \times \mathbf{z})\big) \cdot \mathbf{n}\, dA + \iint_D \rho Q\, dA$$

Since D is an arbitrary domain,

$$\rho \frac{\partial^2 u}{\partial t^2} = \big(\nabla \times T(\mathbf{n} \times \mathbf{z})\big) \cdot \mathbf{n} + \rho Q$$

perfectly elastic membrane: we assume that $T(x, y, t) \approx T_0 = \text{const}$.

Equation of membrane: $H(x, y, z, t) = 0$, where $H(x, y, z, t) = z - u(x, y, t)$.

Normal vector $\mathbf{n}$ is proportional to $\nabla H = (-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1)$.

We assume that $|\nabla H| \approx 1$ so that $\mathbf{n} \approx \nabla H$.

$$\rho \frac{\partial^2 u}{\partial t^2} = T_0\big(\nabla \times (\nabla H \times \mathbf{z})\big) \cdot \nabla H + \rho Q$$

$$\nabla H \times \mathbf{z} = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ -\frac{\partial u}{\partial x} & -\frac{\partial u}{\partial y} & 1 \\ 0 & 0 & 1 \end{vmatrix} = -\frac{\partial u}{\partial y}\,\mathbf{x} + \frac{\partial u}{\partial x}\,\mathbf{y}$$

$$\nabla \times (\nabla H \times \mathbf{z}) = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} & 0 \end{vmatrix} = \big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\big)\mathbf{z}$$

$$\rho \frac{\partial^2 u}{\partial t^2} = T_0\, \nabla^2 u + \rho Q$$

$$\boxed{\rho(x,y)\frac{\partial^2 u}{\partial t^2} = T_0\,\nabla^2 u + \rho(x,y)Q(x,y,t)}$$

Assuming $\rho = \text{const}$ and $Q = 0$, we obtain

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\,\nabla^2 u}$$

where $c^2 = T_0/\rho$.

This is **two-dimensional wave equation**.

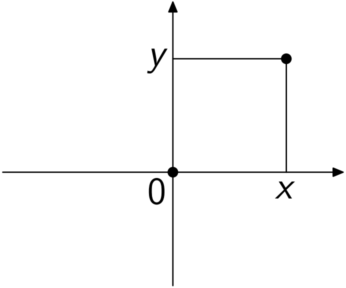
Complex numbers

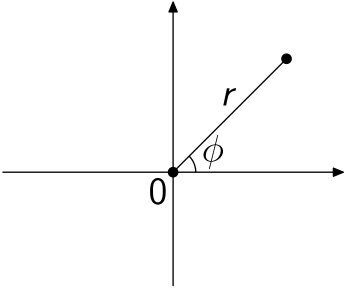
$\mathbb{C}$: complex numbers.
$z = x + iy$, where $x, y \in \mathbb{R}$ and $i^2 = -1$
(that is, $i = \sqrt{-1}$).

x is the **real** part of z,
iy is the **imaginary** part of z.

$z = x + iy$ is identified with the vector $(x, y) \in \mathbb{R}^2$.

$z = r(\cos\phi + i\sin\phi)$, where $r \geq 0$ is the **modulus** of z ($r = |z|$) and $\phi \in \mathbb{R}$ is the **argument** of z (determined up to adding a multiple of 2π).
$|x + iy| = \sqrt{x^2 + y^2}$.





$$z = x + iy$$

$$z = r(\cos\phi + i\sin\phi).$$

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2),$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

If $z_1 = r_1(\cos\phi_1 + i\sin\phi_1)$ and
$z_2 = r_2(\cos\phi_2 + i\sin\phi_2)$, then

$$z_1 z_2 = r_1 r_2(\cos(\phi_1 + \phi_2) + i\sin(\phi_1 + \phi_2)).$$

$e^{i\phi} = \cos\phi + i\sin\phi$ for any $\phi \in \mathbb{R}$.
Then $e^{i(\phi_1+\phi_2)} = e^{i\phi_1} e^{i\phi_2}$, $\phi_1, \phi_2 \in \mathbb{R}$.

$z = re^{i\phi}$, where r is the modulus, ϕ is the argument.

Given $z = x + iy$, the **complex conjugate** of z is $\bar{z} = x - iy$.
The conjugacy $z \mapsto \bar{z}$ is the reflection of $\mathbb{C}$ in the real line.

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$.

$z\bar{z} = |z|^2$, hence $z^{-1} = \dfrac{\bar{z}}{|z|^2}$.

$$(x + iy)^{-1} = \frac{x - iy}{x^2 + y^2}.$$

The set $\mathbb{C}$ of complex numbers is a **field**.

Analytic functions

Suppose $D \subset \mathbb{C}$ is a domain and consider a function $f : D \longrightarrow \mathbb{C}$.
The function f is called **complex differentiable** at a point $z_0 \in D$ if

$$\lim_{z \longrightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists.}$$

The limit value is the **derivative** $f'(z_0)$.

The function f is called **analytic at a point** $z_0 \in D$ if it is complex differentiable in a neighborhood of z_0. f is called **analytic in** D if it is complex differentiable at every point of D.

To a complex function $f : D \to \mathbb{C}$ we associate a real vector-function $(u, v) : D \to \mathbb{R}^2$ defined by $f(x + iy) = u(x, y) + iv(x, y)$.

Theorem The function f is analytic if and only if u, v have continuous partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ and, moreover, the **Cauchy-Riemann** equations are satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Sketch of the proof: f is complex differentiable at z_0 if

$$f(z) = f(z_0) + p \cdot (z - z_0) + \alpha(z),$$

where $p \in \mathbb{C}$ ($p = f'(z_0)$) and $|\alpha(z)|/|z - z_0| \longrightarrow 0$ as $z \longrightarrow z_0$.

(u, v) is differentiable at (x_0, y_0) if

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{pmatrix} + A \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \begin{pmatrix} \beta(x,y) \\ \gamma(x,y) \end{pmatrix},$$

where A is a 2×2 matrix, $A = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$, and

$(\beta(x,y), \gamma(x,y))$ is small when compared with $(x - x_0, y - y_0)$.

When $A = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$ *is the matrix of multiplication?*

Let $p = q + ir$. Then $p \cdot 1 = q + ir$, $p \cdot i = -r + iq$.

It follows that $A = \begin{pmatrix} q & -r \\ r & q \end{pmatrix}$.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The CR equations imply that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}.$$

It follows that $\nabla^2 u = 0$. Similarly, $\nabla^2 v = 0$.

Real and imaginary components of a complex analytic function are **harmonic**.