Math 412-501

Theory of Partial Differential Equations

Lecture 2-3: Separation of variables for the one-dimensional wave equation. Laplace's equation in a rectangle.

Separation of variables: wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Suppose $u(x,t) = \phi(x)G(t)$. Then

$$\frac{\partial^2 u}{\partial t^2} = \phi(x)\frac{d^2 G}{dt^2}, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{d^2 \phi}{dx^2}G(t).$$

Hence

$$\phi(x)\frac{d^2 G}{dt^2} = c^2 \frac{d^2 \phi}{dx^2}G(t).$$

Divide both sides by $c^2 \cdot \phi(x) \cdot G(t) = c^2 \cdot u(x,t)$:

$$\frac{1}{c^2 G} \cdot \frac{d^2 G}{dt^2} = \frac{1}{\phi} \cdot \frac{d^2 \phi}{dx^2}.$$

It follows that

$$\frac{1}{c^2 G} \cdot \frac{d^2 G}{dt^2} = \frac{1}{\phi} \cdot \frac{d^2 \phi}{dx^2} = -\lambda = \text{const.}$$

The variables have been separated:

$$\frac{d^2 \phi}{dx^2} = -\lambda \phi,$$

$$\frac{d^2 G}{dt^2} = -\lambda c^2 G.$$

Proposition Suppose ϕ and G are solutions of the above ODEs for the same value of λ. Then $u(x,t) = \phi(x) G(t)$ is a solution of the wave equation.

Example. $u(x,t) = \cos ct \cdot \sin x$. **(standing wave)**

Finite string with fixed ends

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$
$$u(0,t) = u(L,t) = 0.$$

We are looking for solutions $u(x,t) = \phi(x)G(t)$.

PDE holds if

$$\frac{d^2\phi}{dx^2} = -\lambda\phi,$$
$$\frac{d^2G}{dt^2} = -\lambda c^2 G$$

for the same constant λ.

Boundary conditions hold if

$$\phi(0) = \phi(L) = 0.$$

Eigenvalue problem: $\phi'' = -\lambda\phi$, $\phi(0) = \phi(L) = 0$.

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2$, $n = 1, 2, \dots$

Eigenfunctions: $\phi_n(x) = \sin \frac{n\pi x}{L}$.

Dependence on time:

$$G'' = -\lambda c^2 G$$

$$\implies G(t) = C_1 \cos(c\sqrt{\lambda}t) + C_2 \sin(c\sqrt{\lambda}t)$$

Solution of the heat equation: $u(x,t) = \phi(x)G(t)$.

Theorem For $n = 1, 2, \dots$ and arbitrary constants C_1, C_2, the function

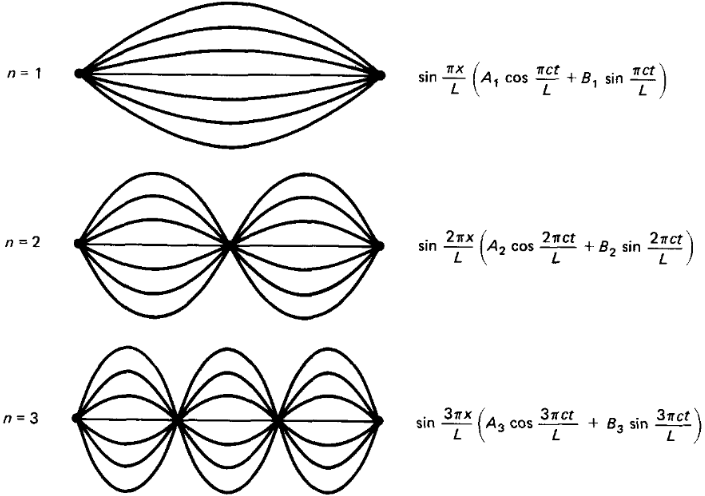
$$u(x,t) = \phi_n(x) \cdot \left(C_1 \cos(c\sqrt{\lambda_n}t) + C_2 \sin(c\sqrt{\lambda_n}t)\right)$$

$$= \sin\frac{n\pi x}{L} \cdot \left(C_1 \cos\frac{n\pi ct}{L} + C_2 \sin\frac{n\pi ct}{L}\right)$$

is a solution of the following boundary value problem for the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(L,t) = 0.$$

Normal modes (a.k.a. harmonics)



Natural frequencies: $nc/(2L)$, $n = 1, 2, \ldots$

Initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$

$$u(x,0) = f(x), \ \ \frac{\partial u}{\partial t}(x,0) = g(x), \qquad u(0,t) = u(L,t) = 0.$$

Principle of superposition: the solution is a superposition of normal modes.

$$u(x,t) = \sum_{n=1}^{\infty} \sin \tfrac{n\pi x}{L} \left(C_n \cos \tfrac{n\pi ct}{L} + D_n \sin \tfrac{n\pi ct}{L} \right)$$

Initial conditions are satisfied if

$$f(x) = \sum_{n=1}^{\infty} C_n \sin \tfrac{n\pi x}{L}$$

$$g(x) = \sum_{n=1}^{\infty} D_n \tfrac{n\pi c}{L} \sin \tfrac{n\pi x}{L}$$

How do we solve the initial-boundary value problem?

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$

$$u(x,0) = f(x), \ \ \frac{\partial u}{\partial t}(x,0) = g(x), \quad u(0,t) = u(L,t) = 0.$$

- Expand f and g into Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \tfrac{n\pi x}{L},$$

$$g(x) = \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi x}{L}.$$

- Write the solution:

$$u(x,t) = \sum_{n=1}^{\infty} \sin \tfrac{n\pi x}{L} \left(C_n \cos \tfrac{n\pi ct}{L} + D_n \sin \tfrac{n\pi ct}{L} \right),$$

where $C_n = a_n$, $D_n = \frac{L}{n\pi c} b_n$.

The solution

$$u(x,t) = \sum_{n=1}^{\infty} \sin \tfrac{n\pi x}{L} \big(C_n \cos \tfrac{n\pi ct}{L} + D_n \sin \tfrac{n\pi ct}{L} \big)$$

is defined in the whole plane.

It satisfies initial conditions

$$u(x,0) = F(x), \;\; \frac{\partial u}{\partial t}(x,0) = G(x), \;\; -\infty < x < \infty,$$

where F and G are the sums of Fourier sine series of f and g, respectively.

F and G are odd $2L$-periodic extensions of f and g.

F and G are odd with respect to 0 and L.

Separation of variables: Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Suppose $u(x,y) = \phi(x)h(y)$. Then

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2\phi}{dx^2}h(y), \qquad \frac{\partial^2 u}{\partial y^2} = \phi(x)\frac{d^2 h}{dy^2}.$$

Hence

$$\frac{d^2\phi}{dx^2}h(y) + \phi(x)\frac{d^2 h}{dy^2} = 0.$$

Divide both sides by $\phi(x)h(y) = u(x,y)$:

$$\frac{1}{\phi}\cdot\frac{d^2\phi}{dx^2} = -\frac{1}{h}\cdot\frac{d^2 h}{dy^2}.$$

It follows that

$$\frac{1}{\phi} \cdot \frac{d^2\phi}{dx^2} = -\frac{1}{h} \cdot \frac{d^2h}{dy^2} = -\lambda = \text{const.}$$

The variables have been separated:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi,$$

$$\frac{d^2h}{dy^2} = \lambda h.$$

Proposition Suppose ϕ and h are solutions of the above ODEs for the same value of λ. Then $u(x,t) = \phi(x)h(y)$ is a solution of Laplace's equation.

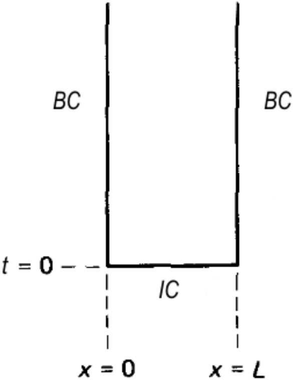
Example. $u(x,y) = e^y \sin x$.

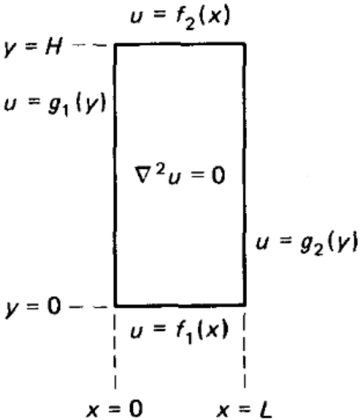
Laplace's equation inside a rectangle

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (0 < x < L,\ 0 < y < H)$$

Boundary conditions:

$$\begin{aligned} u(0, y) &= g_1(y) \\ u(L, y) &= g_2(y) \\ u(x, 0) &= f_1(x) \\ u(x, H) &= f_2(x) \end{aligned}$$


BC
BC
$t = 0$
IC
$x = 0$
$x = L$


$u = f_2(x)$
$y = H$
$u = g_1(y)$
$\nabla^2 u = 0$
$u = g_2(y)$
$y = 0$
$u = f_1(x)$
$x = 0$
$x = L$

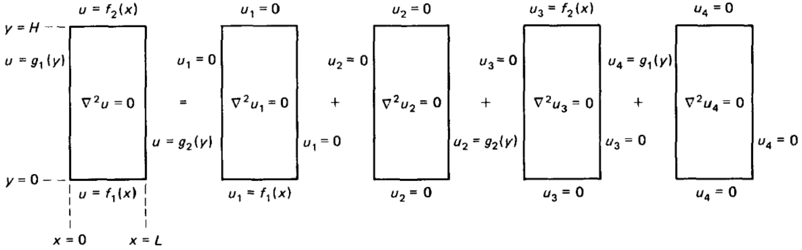
Principle of superposition:

$$u = u_1 + u_2 + u_3 + u_4,$$

where

$$\nabla^2 u_1 = \nabla^2 u_2 = \nabla^2 u_3 = \nabla^2 u_4 = 0,$$

$$u_1(x,0) = f_1(x), \quad u_1(0,y) = u_1(L,y) = u_1(x,H) = 0;$$
$$u_2(L,y) = g_2(y), \quad u_2(0,y) = u_2(x,0) = u_2(x,H) = 0;$$
$$u_3(x,H) = f_2(x), \quad u_3(0,y) = u_3(L,y) = u_3(x,0) = 0;$$
$$u_4(0,y) = g_1(y), \quad u_4(L,y) = u_4(x,0) = u_4(x,H) = 0.$$



u = f2(x)
y = H
u = g1(y)
∇2u = 0
u = g2(y)
y = 0
u = f1(x)
x = 0
x = L
=
u1 = 0
u1 = 0
∇2u1 = 0
u1 = 0
u1 = f1(x)
+
u2 = 0
u2 = 0
∇2u2 = 0
u2 = g2(y)
u2 = 0
+
u3 = f2(x)
u3 = 0
∇2u3 = 0
u3 = 0
u3 = 0
+
u4 = 0
u4 = g1(y)
∇2u4 = 0
u4 = 0
u4 = 0

Reduced boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (0 < x < L,\ 0 < y < H)$$

Boundary conditions:

$$\begin{aligned} u(0, y) &= 0 \\ u(L, y) &= 0 \\ u(x, 0) &= f_1(x) \\ u(x, H) &= 0 \end{aligned}$$

Separation of variables

We are looking for a solution $u(x, y) = \phi(x)h(y)$.

PDE holds if

$$\frac{d^2\phi}{dx^2} = -\lambda\phi,$$

$$\frac{d^2h}{dy^2} = \lambda h$$

for the same constant λ.

Boundary conditions $u(0, y) = u(L, y) = 0$ hold if

$$\phi(0) = \phi(L) = 0.$$

Boundary condition $u(x, H) = 0$ holds if

$$h(H) = 0.$$

Eigenvalue problem: $\phi'' = -\lambda\phi, \;\; \phi(0) = \phi(L) = 0.$

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2$, $n = 1, 2, \ldots$

Eigenfunctions: $\phi_n(x) = \sin\frac{n\pi x}{L}$.

Dependence on y:

$$h'' = \lambda h, \qquad h(H) = 0.$$

$$\Longrightarrow \; h(y) = C_0 \sinh\sqrt{\lambda}(y - H)$$

Solution of Laplace's equation:

$$u(x, y) = \sin\frac{n\pi x}{L}\sinh\frac{n\pi(y-H)}{L}, \qquad n = 1, 2, \ldots$$

We are looking for the solution of the reduced boundary value problem as a superposition of solutions with separated variables.

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi(y-H)}{L}$$

Boundary condition $u(x,0) = f_1(x)$ is satisfied if

$$f(x) = -\sum_{n=1}^{\infty} C_n \sinh \frac{n\pi H}{L} \sin \frac{n\pi x}{L}$$

How do we solve the reduced boundary value problem?

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (0 < x < L,\ 0 < y < H),$$

$$u(x,0) = f_1(x),\ \ u(x,H) = u(0,y) = u(L,y) = 0.$$

- Expand f_1 into the Fourier sine series:

$$f_1(x) = \sum_{n=1}^{\infty} a_n \sin \tfrac{n\pi x}{L}.$$

- Write the solution:

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin \tfrac{n\pi x}{L} \sinh \tfrac{n\pi(y-H)}{L},$$

where $C_n = -\dfrac{a_n}{\sinh \frac{n\pi H}{L}}$.