Math 412-501

Theory of Partial Differential Equations

Lecture 2-6: Heat and wave equations in box-shaped regions.

Heat conduction in a rectangle

Initial-boundary value problem:

$$\frac{\partial u}{\partial t} = k\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) \quad (0 < x < L,\ 0 < y < H),$$

$$u(x, y, 0) = f(x, y) \quad (0 < x < L,\ 0 < y < H),$$

$$u(0, y, t) = u(L, y, t) = 0 \quad (0 < y < H),$$

$$u(x, 0, t) = u(x, H, t) = 0 \quad (0 < x < L).$$

We are looking for solutions to the boundary value problem with separated variables.

Separation of variables: $u(x,y,t) = \phi(x)h(y)G(t)$.
Substitute this into the heat equation:

$$\phi(x)h(y)\frac{dG}{dt} = k\left(\frac{d^2\phi}{dx^2}h(y)G(t) + \phi(x)\frac{d^2h}{dy^2}G(t)\right).$$

Divide both sides by
$k \cdot \phi(x)h(y)G(t) = k \cdot u(x,y,t)$:

$$\frac{1}{kG}\cdot\frac{dG}{dt} = \frac{1}{\phi}\cdot\frac{d^2\phi}{dx^2} + \frac{1}{h}\cdot\frac{d^2h}{dy^2}.$$

It follows that $\dfrac{1}{\phi}\cdot\dfrac{d^2\phi}{dx^2} = -\lambda$, $\dfrac{1}{h}\cdot\dfrac{d^2h}{dy^2} = -\mu$,
$\dfrac{1}{kG}\cdot\dfrac{dG}{dt} = -\lambda - \mu$, where λ and μ are **separation constants**.

The variables have been separated:

$$\frac{dG}{dt} = -(\lambda + \mu)kG,$$

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \qquad \frac{d^2h}{dy^2} = -\mu h.$$

Proposition Suppose G, ϕ, and h are solutions of the above ODEs for the same values of λ and μ. Then $u(x, y, t) = \phi(x)h(y)G(t)$ is a solution of the heat equation.

Boundary conditions $u(0, y, t) = u(L, y, t) = 0$ hold if $\phi(0) = \phi(L) = 0$.

Boundary conditions $u(x, 0, t) = u(x, H, t) = 0$ hold if $h(0) = h(H) = 0$.

1st eigenvalue problem: $\phi'' = -\lambda\phi,\ \ \phi(0) = \phi(L) = 0.$

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2,\ n = 1, 2, \ldots$

Eigenfunctions: $\phi_n(x) = \sin\frac{n\pi x}{L}.$

2nd eigenvalue problem: $h'' = -\mu h,\ \ h(0) = h(H) = 0.$

Eigenvalues: $\mu_m = (\frac{m\pi}{H})^2,\ m = 1, 2, \ldots$

Eigenfunctions: $h_m(y) = \sin\frac{m\pi y}{H}.$

Dependence on t:

$$G'(t) = -(\lambda + \mu)kG(t) \implies G(t) = C_0 e^{-(\lambda+\mu)kt}$$

Solution of the boundary value problem:

$$u(x, y, t) = e^{-(\lambda_n+\mu_m)kt}\phi_n(x)h_m(y)$$

$$= \exp\Big(-((\tfrac{n\pi}{L})^2 + (\tfrac{m\pi}{H})^2)kt\Big) \cdot \sin\tfrac{n\pi x}{L} \cdot \sin\tfrac{m\pi y}{H}.$$

We are looking for the solution of the initial-boundary value problem as a superposition of solutions with separated variables.

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{n,m} e^{-(\lambda_n+\mu_m)kt} \phi_n(x) h_m(y)$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{n,m} \exp\Big(-\big(\big(\tfrac{n\pi}{L}\big)^2 + \big(\tfrac{m\pi}{H}\big)^2\big)kt\Big) \cdot \sin \tfrac{n\pi x}{L} \cdot \sin \tfrac{m\pi y}{H}$$

How do we find coefficients $C_{n,m}$?
From the initial condition $u(x,y,0) = f(x,y)$.

$$f(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{n,m} \sin \tfrac{n\pi x}{L} \sin \tfrac{m\pi y}{H}$$

(double Fourier sine series)

How do we solve the heat conduction problem?

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (0 < x < L,\ 0 < y < H),$$

$$u(x, y, 0) = f(x, y) \quad (0 < x < L,\ 0 < y < H),$$

$$u(0, y, t) = u(L, y, t) = 0 \quad (0 < y < H),$$

$$u(x, 0, t) = u(x, H, t) = 0 \quad (0 < x < L).$$

- Expand f into the double Fourier sine series:

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{n,m} \sin \tfrac{n\pi x}{L} \sin \tfrac{m\pi y}{H}.$$

- Write the solution: $u(x, y, t) =$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{n,m} \exp\left(-\left(\left(\tfrac{n\pi}{L}\right)^2 + \left(\tfrac{m\pi}{H}\right)^2\right)kt\right) \sin \tfrac{n\pi x}{L} \sin \tfrac{m\pi y}{H}.$$

How do we expand f into the double Fourier sine series?

$$f(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{n,m} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}.$$

For simplicity, assume that f is smooth and vanishes on the boundary of the rectangle $[0, L] \times [0, H]$.

Fix any $y \in [0, H]$. Then $g(x) = f(x, y)$ is a smooth function on $[0, L]$ and $g(0) = g(L) = 0$.

Hence $g(x)$ can be expanded into the Fourier sine series on $[0, L]$:

$$g(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}.$$

Note that each B_n depends on y; $B_n = B_n(y)$.

$$f(x,y) = \sum_{n=1}^{\infty} B_n(y) \sin \frac{n\pi x}{L}$$

Can we expand $B_n(y)$ into the Fourier sine series on $[0, H]$?

$$B_n(y) = \frac{2}{L} \int_0^L f(x,y) \sin \frac{n\pi x}{L}\, dx$$

It follows that $B_n(y)$ is smooth and $B_n(0) = B_n(H) = 0$.

$$B_n(y) = \sum_{m=1}^{\infty} b_{n,m} \sin \frac{m\pi y}{H}$$

$$f(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{n,m} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}$$

$$b_{n,m} = \frac{2}{H}\int_0^H B_n(y)\sin\frac{m\pi y}{H}\,dy$$

$$= \frac{2}{H}\int_0^H \left(\frac{2}{L}\int_0^L f(x,y)\sin\frac{n\pi x}{L}\,dx\right)\sin\frac{m\pi y}{H}\,dy$$

$$= \frac{4}{LH}\int_0^L\int_0^H f(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\,dx\,dy.$$

How do we solve the heat conduction problem?

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (0 < x < L,\ 0 < y < H),$$

$$u(x, y, 0) = f(x, y) \quad (0 < x < L,\ 0 < y < H),$$

$$u(0, y, t) = u(L, y, t) = 0 \quad (0 < y < H),$$

$$u(x, 0, t) = u(x, H, t) = 0 \quad (0 < x < L).$$

Solution: $u(x, y, t) =$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{n,m} \exp\left(-((\tfrac{n\pi}{L})^2 + (\tfrac{m\pi}{H})^2)kt\right) \sin\tfrac{n\pi x}{L} \sin\tfrac{m\pi y}{H},$$

where

$$b_{n,m} = \frac{4}{LH}\int_0^L\int_0^H f(x, y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\,dx\,dy.$$

Vibrating rectangular membrane

Initial-boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (0 < x < L,\ 0 < y < H),$$

$$u(x, y, 0) = f(x, y),$$
$$\frac{\partial u}{\partial t}(x, y, 0) = g(x, y) \quad (0 < x < L,\ 0 < y < H),$$
$$u(0, y, t) = u(L, y, t) = 0 \quad (0 < y < H),$$
$$u(x, 0, t) = u(x, H, t) = 0 \quad (0 < x < L).$$

We are looking for solutions to the boundary value problem with separated variables.

Separation of variables: $u(x, y, t) = \phi(x)h(y)G(t)$.
Substitute this into the wave equation:

$$\phi(x)h(y)\frac{d^2G}{dt^2} = c^2\left(\frac{d^2\phi}{dx^2}h(y)G(t) + \phi(x)\frac{d^2h}{dy^2}G(t)\right).$$

Divide both sides by $c^2\phi(x)h(y)G(t) = c^2u(x, y, t)$:

$$\frac{1}{c^2G}\cdot\frac{d^2G}{dt^2} = \frac{1}{\phi}\cdot\frac{d^2\phi}{dx^2} + \frac{1}{h}\cdot\frac{d^2h}{dy^2}.$$

It follows that $\frac{1}{\phi}\cdot\frac{d^2\phi}{dx^2} = -\lambda$, $\frac{1}{h}\cdot\frac{d^2h}{dy^2} = -\mu$,
$\frac{1}{c^2G}\cdot\frac{d^2G}{dt^2} = -\lambda - \mu$, where λ and μ are separation constants.

The variables have been separated:

$$\frac{d^2G}{dt^2} = -(\lambda + \mu)c^2 G,$$

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \qquad \frac{d^2h}{dy^2} = -\mu h.$$

Proposition Suppose G, ϕ, and h are solutions of the above ODEs for the same values of λ and μ. Then $u(x, y, t) = \phi(x)h(y)G(t)$ is a solution of the heat equation.

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1st eigenvalue problem: $\phi'' = -\lambda\phi$, $\phi(0) = \phi(L) = 0$.

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2$, $n = 1, 2, \ldots$

Eigenfunctions: $\phi_n(x) = \sin\frac{n\pi x}{L}$.

2nd eigenvalue problem: $h'' = -\mu h$, $h(0) = h(H) = 0$.

Eigenvalues: $\mu_m = (\frac{m\pi}{H})^2$, $m = 1, 2, \ldots$

Eigenfunctions: $h_m(y) = \sin\frac{m\pi y}{H}$.

Dependence on t: $\quad G''(t) = -(\lambda + \mu)c^2 G(t)$

$$\Longrightarrow \; G(t) = C_1 \cos(\sqrt{\lambda + \mu} \cdot ct) + C_2 \sin(\sqrt{\lambda + \mu} \cdot ct)$$

Solution of the boundary value problem:

$$u(x, y, t) =$$

$$= \big(C_1 \cos(\sqrt{\lambda_n + \mu_m}\, ct) + C_2 \sin(\sqrt{\lambda_n + \mu_m}\, ct)\big)\phi_n(x) h_m(y).$$

We are looking for the solution of the initial-boundary value problem as a superposition of solutions with separated variables.

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\big(C_{n,m}\cos(\sqrt{\lambda_n+\mu_m}\,ct) + D_{n,m}\sin(\sqrt{\lambda_n+\mu_m}\,ct)\big)\sin\tfrac{n\pi x}{L}\sin\tfrac{m\pi y}{H}$$

How do we find coefficients $C_{n,m}$ and $D_{n,m}$?
From initial conditions $u(x,y,0) = f(x,y)$ and $\frac{\partial u}{\partial t}(x,y,0) = g(x,y)$.

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\sin\tfrac{n\pi x}{L}\sin\tfrac{m\pi y}{H}$$

$$g(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sqrt{\lambda_n+\mu_m}\,cD_{n,m}\sin\tfrac{n\pi x}{L}\sin\tfrac{m\pi y}{H}$$

How do we solve the initial-boundary value problem?

$$\frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) \quad (0 < x < L,\, 0 < y < H),$$

$$u(x, y, 0) = f(x, y),$$
$$\tfrac{\partial u}{\partial t}(x, y, 0) = g(x, y) \quad (0 < x < L,\, 0 < y < H),$$
$$u(0, y, t) = u(L, y, t) = 0 \quad (0 < y < H),$$
$$u(x, 0, t) = u(x, H, t) = 0 \quad (0 < x < L).$$

- Expand f and g into the double Fourier sine series:

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} \sin \tfrac{n\pi x}{L} \sin \tfrac{m\pi y}{H},$$

$$g(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{n,m} \sin \tfrac{n\pi x}{L} \sin \tfrac{m\pi y}{H}.$$

- Write the solution:

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Bigg(a_{n,m}\cos(\sqrt{\lambda_n+\mu_m}\,ct)$$

$$+\, b_{n,m}\frac{\sin(\sqrt{\lambda_n+\mu_m}\,ct)}{\sqrt{\lambda_n+\mu_m}\,c}\Bigg)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H},$$

where $\lambda_n + \mu_m = (\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2$,

$$a_{n,m} = \frac{4}{LH}\int_0^L\int_0^H f(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\,dx\,dy,$$

$$b_{n,m} = \frac{4}{LH}\int_0^L\int_0^H g(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\,dx\,dy.$$