Math 412-501

Theory of Partial Differential Equations

Lecture 2-7:
Sturm-Liouville eigenvalue problems.

Sturm-Liouville differential equation:

$$\boxed{\frac{d}{dx}\Big(p\frac{d\phi}{dx}\Big) + q\phi + \lambda\sigma\phi = 0 \qquad (a < x < b),}$$

where $p = p(x)$, $q = q(x)$, $\sigma = \sigma(x)$ are known functions on $[a, b]$ and λ is an unknown constant.

The Sturm-Liouville equation is a linear homogeneous ODE of the second order.

Sturm-Liouville eigenvalue problem $=$
$=$ Sturm-Liouville differential equation $+$
$+$ linear homogeneous boundary conditions

J. C. F. Sturm
(1803–1855)

J. Liouville
(1809–1882)

The Sturm-Liouville equation usually arises after separation of variables in a linear homogeneous PDE of the second order.

Examples.

- $\phi'' + \lambda\phi = 0$ (heat, wave, Laplace's equations)
- $r^2 \dfrac{d^2h}{dr^2} + r\,\dfrac{dh}{dr} = \lambda h$

(Laplace's equation in polar coordinates)

standard notation: $x^2\phi'' + x\phi' - \lambda\phi = 0$

canonical form: $(x\phi')' - \lambda x^{-1}\phi = 0$

Heat flow in a nonuniform rod:

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q,$$

$K_0 = K_0(x)$, $c = c(x)$, $\rho = \rho(x)$, $Q = Q(u, x, t)$.

The equation is linear homogeneous if $Q = \alpha(x, t)u$.
We assume that $\alpha = \alpha(x)$.

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + \alpha u$$

Separation of variables: $u(x, t) = \phi(x)G(t)$.
Substitute this into the heat equation:

$$c\rho\phi\frac{dG}{dt} = \frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right)G + \alpha\phi G.$$

Divide both sides by $c(x)\rho(x)\phi(x)G(t) = c\rho u$:

$$\frac{1}{G}\,\frac{dG}{dt} = \frac{1}{c\rho\phi}\,\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \frac{\alpha}{c\rho} = -\lambda = \text{const.}$$

The variables have been separated:

$$\frac{dG}{dt} + \lambda G = 0,$$

$$\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \alpha\phi + \lambda c\rho\phi = 0.$$

Sturm-Liouville differential equation:

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0 \quad (a < x < b).$$

Examples of boundary conditions:

- $\phi(a) = \phi(b) = 0$ (Dirichlet conditions)
- $\phi'(a) = \phi'(b) = 0$ (von Neumann conditions)
- $\phi'(a) = 2\phi(a)$, $\phi'(b) = -3\phi(b)$ (Robin conditions)
- $\phi(a) = 0$, $\phi'(b) = 0$ (mixed conditions)
- $\phi(a) = \phi(b)$, $\phi'(a) = \phi'(b)$ (periodic conditions)
- $|\phi(a)| < \infty$, $\phi(b) = 0$ (singular conditions)

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0 \qquad (a < x < b).$$

The equation is **regular** if p, q, σ are real and continuous on $[a, b]$, and $p, \sigma > 0$ on $[a, b]$.

The Sturm-Liouville eigenvalue problem is **regular** if the equation is regular and boundary conditions are of the form

$$\beta_1\phi(a) + \beta_2\phi'(a) = 0,$$
$$\beta_3\phi(b) + \beta_4\phi'(b) = 0,$$

where $\beta_i \in \mathbb{R}$, $|\beta_1| + |\beta_2| \neq 0$, $|\beta_3| + |\beta_4| \neq 0$.

This includes Dirichlet, Neumann, and Robin conditions but excludes periodic and singular ones.

Regular Sturm-Liouville eigenvalue problem:

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0 \qquad (a < x < b),$$

$$\beta_1\phi(a) + \beta_2\phi'(a) = 0,$$
$$\beta_3\phi(b) + \beta_4\phi'(b) = 0.$$

Eigenfunction: nonzero solution ϕ of the boundary value problem.

Eigenvalue: corresponding value of λ.

Eigenvalues and eigenfunctions of a regular Sturm-Liouville eigenvalue problem have **six** important properties.

Property 1. All eigenvalues are real.

Property 2. All eigenvalues can be arranged in the ascending order

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots$$

so that $\lambda_n \to \infty$ as $n \to \infty$.

This means that:

- there are infinitely many eigenvalues;
- there is a smallest eigenvalue;
- on any finite interval, there are only finitely many eigenvalues.

Remark. It is possible that $\lambda_1 < 0$.

Property 3. Given an eigenvalue λ_n, the corresponding eigenfunction ϕ_n is unique up to a multiplicative constant. The function ϕ_n has exactly $n-1$ zeros in (a, b).

We say that λ_n is a **simple** eigenvalue.

Property 4. Eigenfunctions belonging to different eigenvalues satisfy an integral identity:

$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0 \quad \text{if} \quad \lambda_n \neq \lambda_m.$$

We say that ϕ_n and ϕ_m are **orthogonal** relative to the weight function σ.

Property 5. Any eigenvalue λ can be related to its eigenfunction ϕ as follows:

$$\lambda = \frac{-p\phi\phi' \Big|_a^b + \int_a^b \left(p(\phi')^2 - q\phi^2\right) dx}{\int_a^b \phi^2 \sigma \, dx}.$$

The right-hand side is called the **Rayleigh quotient**.

Property 6. Any piecewise continuous function $f : [a, b] \to \mathbb{R}$ is assigned a series

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x),$$

where

$$c_n = \frac{\displaystyle\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\displaystyle\int_a^b \phi_n^2(x)\sigma(x)\,dx}.$$

If f is piecewise smooth then the series converges for any $a < x < b$. The sum is equal to $f(x)$ if f is continuous at x. Otherwise the series converges to $\frac{1}{2}(f(x+) + f(x-))$.

We say that the set of eigenfunctions ϕ_n is **complete**.

A regular Sturm-Liouville eigenvalue problem:

$$\phi'' + \lambda\phi = 0, \qquad \phi(0) = \phi(L) = 0.$$

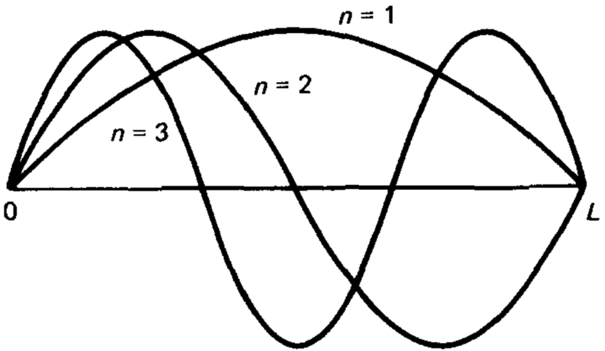
$(p = \sigma = 1,\ q = 0,\ [a, b] = [0, L])$

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2$, $n = 1, 2, \ldots$

Eigenfunctions: $\phi_n(x) = \sin \frac{n\pi x}{L}$.

The zeros of ϕ_n divide the interval $[0, L]$ into n equal parts.

Property 3a. Suppose $x_1 < x_2 < \ldots < x_{n-1}$ are zeros of the eigenfunction ϕ_n in (a, b). Then ϕ_{n+1} has exactly one zero in each of the following intervals: $(a, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-2}, x_{n-1})$, (x_{n-1}, b).



Eigenfunctions ϕ_n

Orthogonality: $\displaystyle\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = 0, \quad n \neq m.$

Rayleigh quotient: $$\lambda = \frac{\displaystyle\int_0^L |\phi'(x)|^2\,dx}{\displaystyle\int_0^L |\phi(x)|^2\,dx}.$$

Fourier sine series: $f \sim \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}$,

$$\text{where} \quad c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Note that $\displaystyle\int_0^L \left(\sin\frac{n\pi x}{L}\right)^2 dx = \frac{L}{2}.$