Math 412-501

Theory of Partial Differential Equations

Lecture 2-8:
Sturm-Liouville eigenvalue problems (continued).

Sturm-Liouville differential equation:

$$\boxed{\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0 \qquad (a < x < b),}$$

where $p = p(x)$, $q = q(x)$, $\sigma = \sigma(x)$ are known functions on $[a, b]$ and λ is an unknown constant.

Sturm-Liouville eigenvalue problem $=$
$=$ Sturm-Liouville differential equation $+$
$+$ linear homogeneous boundary conditions

Eigenfunction: nonzero solution ϕ of the boundary value problem.

Eigenvalue: corresponding value of λ.

$$\frac{d}{dx}\Big(p\frac{d\phi}{dx}\Big) + q\phi + \lambda\sigma\phi = 0 \qquad (a < x < b).$$

The equation is **regular** if p, q, σ are real and continuous on $[a, b]$, and $p, \sigma > 0$ on $[a, b]$.

The Sturm-Liouville eigenvalue problem is **regular** if the equation is regular and boundary conditions are of the form

$$\beta_1\phi(a) + \beta_2\phi'(a) = 0,$$
$$\beta_3\phi(b) + \beta_4\phi'(b) = 0,$$

where $\beta_i \in \mathbb{R}$, $|\beta_1| + |\beta_2| \neq 0$, $|\beta_3| + |\beta_4| \neq 0$.

6 properties of a regular Sturm-Liouville problem

- Eigenvalues are real.
- Eigenvalues form an increasing sequence.
- n-th eigenfunction has $n-1$ zeros in (a, b).
- Eigenfunctions are orthogonal with weight σ.
- Eigenfunctions and eigenvalues are related through the Rayleigh quotient.
- Piecewise smooth functions can be expanded into generalized Fourier series of eigenfunctions.

Heat flow in a nonuniform rod without sources

Initial-boundary value problem:

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) \quad (0 < x < L),$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, \text{ (insulated ends)}$$

$$u(x,0) = f(x) \ \ (0 < x < L).$$

We assume that $K_0(x)$, $c(x)$, $\rho(x)$ are positive and continuous on $[0, L]$. Also, we assume that $f(x)$ is piecewise smooth.

Separation of variables: $u(x,t) = \phi(x)G(t)$.
Substitute this into the heat equation:

$$c\rho\phi\frac{dG}{dt} = \frac{d}{dx}\Big(K_0\frac{d\phi}{dx}\Big)G.$$

Divide both sides by $c(x)\rho(x)\phi(x)G(t) = c\rho u$:

$$\frac{1}{G}\,\frac{dG}{dt} = \frac{1}{c\rho\phi}\,\frac{d}{dx}\Big(K_0\frac{d\phi}{dx}\Big) = -\lambda = \text{const.}$$

The variables have been separated:

$$\frac{dG}{dt} + \lambda G = 0, \qquad \frac{d}{dx}\Big(K_0\frac{d\phi}{dx}\Big) + \lambda c\rho\phi = 0.$$

Boundary conditions $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$ hold provided $\phi'(0) = \phi'(L) = 0$.

Eigenvalue problem:

$$\frac{d}{dx}\Big(K_0 \frac{d\phi}{dx}\Big) + \lambda c\rho\phi = 0, \qquad \phi'(0) = \phi'(L) = 0.$$

This is a regular Sturm-Liouville eigenvalue problem ($p = K_0$, $q = 0$, $\sigma = c\rho$, $[a,b] = [0,L]$).

There are infinitely many eigenvalues:

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots$$

The corresponding eigenfunctions ϕ_n are unique up to multiplicative constants.

Dependence on t:

$$G'(t) = -\lambda G(t) \implies G(t) = C_0 e^{-\lambda t}$$

Solutions of the boundary value problem:

$$u(x,t) = e^{-\lambda_n t}\phi_n(x), \quad n = 1, 2, \dots$$

The general solution of the boundary value problem is a superposition of solutions with separated variables:

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n t}\phi_n(x).$$

Initial condition $u(x,0) = f(x)$ is satisfied when

$$f(x) = \sum_{n=1}^{\infty} C_n\phi_n(x).$$

Hence C_n are coefficients of the generalized Fourier series for f:

$$C_n = \frac{\int_0^L f(x)\phi_n(x)c(x)\rho(x)\,dx}{\int_0^L \phi_n^2(x)c(x)\rho(x)\,dx}.$$

Solution: $u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n t}\phi_n(x).$

In general, we do not know λ_n and ϕ_n.
Nevertheless, we can determine $\lim_{t\to+\infty} u(x,t)$.
We need to know which λ_n is >0, $=0$, <0.

$$\frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right) + \lambda c \rho \phi = 0, \quad \phi'(0) = \phi'(L) = 0.$$

Rayleigh quotient:

$$\lambda = \frac{-K_0 \phi \phi' \Big|_0^L + \int_0^L K_0 (\phi')^2 \, dx}{\int_0^L \phi^2 c \rho \, dx}.$$

Since $\phi'(0) = \phi'(L) = 0$, the nonintegral term vanishes. It follows that either $\lambda > 0$, or else $\lambda = 0$ and $\phi = \text{const}$. Indeed, $\lambda = 0$ is an eigenvalue.

Solution of the heat conduction problem:

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n t} \phi_n(x).$$

Now we know that $\lambda_1 = 0$. Furthermore, we can set $\phi_1 = 1$. Besides, $0 < \lambda_2 < \lambda_3 < \dots$

It follows that

$$\lim_{t \to +\infty} u(x,t) = C_1 = \frac{\displaystyle\int_0^L f(x)c(x)\rho(x)\,dx}{\displaystyle\int_0^L c(x)\rho(x)\,dx}.$$

Rayleigh quotient

Consider a regular Sturm-Liouville equation:

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0 \qquad (a < x < b).$$

Suppose ϕ is a nonzero solution for some λ.
Multiply the equation by ϕ and integrate over $[a, b]$:

$$\int_a^b \phi\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) dx + \int_a^b q\phi^2\, dx + \lambda\int_a^b \sigma\phi^2\, dx = 0.$$

Integrate the first integral by parts:

$$\int_a^b \phi\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) dx = p\phi\frac{d\phi}{dx}\Big|_a^b - \int_a^b p\left(\frac{d\phi}{dx}\right)^2 dx.$$

It follows that

$$\lambda = \frac{-p\phi\phi' \Big|_a^b + \int_a^b \left(p(\phi')^2 - q\phi^2\right) dx}{\int_a^b \phi^2 \sigma \, dx}.$$

We have used only the facts that p, q, σ are continuous and that $\sigma > 0$.

The Rayleigh quotient can be used for **any** boundary conditions.

Regular Sturm-Liouville equation:

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0 \qquad (a < x < b).$$

Consider a linear differential operator

$$\mathcal{L}(f) = \frac{d}{dx}\left(p\frac{df}{dx}\right) + qf.$$

Now the equation can be rewritten as

$$\mathcal{L}(\phi) + \lambda\sigma\phi = 0.$$

Lemma Suppose f and g are functions on $[a, b]$ such that $\mathcal{L}(f)$ and $\mathcal{L}(g)$ are well defined. Then

$$g\mathcal{L}(f) - f\mathcal{L}(g) = \frac{d}{dx}\Big(p(gf' - fg')\Big).$$

Proof: $\mathcal{L}(f) = (pf')' + qf, \ \ \mathcal{L}(g) = (pg')' + qg.$

Left-hand side:

$$g\mathcal{L}(f) - f\mathcal{L}(g) = g(pf')' + gqf - f(pg')' - fqg$$
$$= g(pf')' - f(pg')'.$$

Right-hand side:

$$\frac{d}{dx}\Big(p(gf' - fg')\Big) = \frac{d}{dx}\Big(g(pf') - f(pg')\Big)$$

$$= g'pf' + g(pf')' - f'pg' - f(pg')'$$

$$= g(pf')' - f(pg')'.$$

Lagrange's identity:

$$g\mathcal{L}(f) - f\mathcal{L}(g) = \frac{d}{dx}\Big(p(gf' - fg')\Big)$$

Integrating over $[a, b]$, we obtain **Green's formula**:

$$\int_a^b \Big(g\mathcal{L}(f) - f\mathcal{L}(g)\Big)\, dx = p(gf' - fg') \Big|_a^b$$

Claim If f and g satisfy the same regular boundary conditions, then the right-hand side in Green's formula vanishes.

Proof: We have that

$$\beta_1 f(a) + \beta_2 f'(a) = 0, \qquad \beta_1 g(a) + \beta_2 g'(a) = 0,$$

where $\beta_1, \beta_2 \in \mathbb{R}$, $|\beta_1| + |\beta_2| \neq 0$.

Vectors $(f(a), f'(a))$ and $(g(a), g'(a))$ are orthogonal to vector (β_1, β_2). Since $(\beta_1, \beta_2) \neq 0$, it follows that $(f(a), f'(a))$ and $(g(a), g'(a))$ are parallel. Then their vector product is equal to 0:

$$(g(a), g'(a)) \times (f(a), f'(a)) = g(a)f'(a) - f(a)g'(a) = 0.$$

Similarly, $g(b)f'(b) - f(b)g'(b) = 0$.
Hence

$$p(gf' - fg') \Big|_a^b = 0.$$