Math 412-501

Theory of Partial Differential Equations

Lecture 2-9:
Sturm-Liouville eigenvalue problems (continued).

Regular Sturm-Liouville eigenvalue problem:

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0 \qquad (a < x < b),$$

$$\beta_1\phi(a) + \beta_2\phi'(a) = 0,$$

$$\beta_3\phi(b) + \beta_4\phi'(b) = 0.$$

Here $\beta_i \in \mathbb{R}$, $|\beta_1| + |\beta_2| \neq 0$, $|\beta_3| + |\beta_4| \neq 0$.
Functions p, q, σ are continuous on $[a, b]$,
$p > 0$ and $\sigma > 0$ on $[a, b]$.

6 properties of a regular Sturm-Liouville problem

- Eigenvalues are real.
- Eigenvalues form an increasing sequence.
- n-th eigenfunction has $n-1$ zeros in (a, b).
- Eigenfunctions are orthogonal with weight σ.
- Eigenfunctions and eigenvalues are related through the Rayleigh quotient.
- Piecewise smooth functions can be expanded into generalized Fourier series of eigenfunctions.

Regular Sturm-Liouville equation:

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0 \qquad (a < x < b).$$

Consider a linear differential operator

$$\mathcal{L}(f) = \frac{d}{dx}\left(p\frac{df}{dx}\right) + qf.$$

Now the equation can be rewritten as

$$\mathcal{L}(\phi) + \lambda\sigma\phi = 0.$$

Lagrange's identity:

$$g\mathcal{L}(f) - f\mathcal{L}(g) = \frac{d}{dx}\Big(p(gf' - fg')\Big)$$

Integrating over $[a, b]$, we obtain **Green's formula**:

$$\int_a^b \Big(g\mathcal{L}(f) - f\mathcal{L}(g)\Big)\,dx = p(gf' - fg')\,\Big|_a^b$$

Claim If f and g satisfy the same regular boundary conditions, then the right-hand side in Green's formula vanishes.

Suppose ϕ_n and ϕ_m are eigenfunctions of the Sturm-Liouville problem corresponding to eigenvalues λ_n and λ_m:

$$\mathcal{L}(\phi_n) + \lambda_n \sigma \phi_n = 0, \quad \mathcal{L}(\phi_m) + \lambda_m \sigma \phi_m = 0.$$

Since ϕ_n and ϕ_m satisfy the same regular boundary conditions, Green's formula implies that

$$\int_a^b \Big(\phi_m \mathcal{L}(\phi_n) - \phi_n \mathcal{L}(\phi_m) \Big) \, dx = 0$$

$$\implies \int_a^b (\lambda_m - \lambda_n) \phi_n(x) \phi_m(x) \sigma(x) \, dx = 0$$

If $\lambda_n \neq \lambda_m$, then $\displaystyle\int_a^b \phi_n(x) \phi_m(x) \sigma(x) \, dx = 0.$

Suppose ϕ is a complex-valued eigenfunction corresponding to a complex eigenvalue λ:

$$\mathcal{L}(\phi) + \lambda\sigma\phi = 0,$$

$$\beta_1\phi(a) + \beta_2\phi'(a) = 0,$$
$$\beta_3\phi(b) + \beta_4\phi'(b) = 0.$$

We are going to show that $\lambda \in \mathbb{R}$.

Any complex number $z = x + iy$ is assigned its **complex conjugate** $\bar{z} = x - iy$.

Let us apply the complex conjugacy to the Sturm-liouville equation and the boundary conditions.

$$\overline{\mathcal{L}(\phi) + \lambda\sigma\phi} = 0,$$
$$\overline{\beta_1\phi(a) + \beta_2\phi'(a)} = \overline{\beta_3\phi(b) + \beta_4\phi'(b)} = 0.$$

It is known that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$.

$$\overline{\mathcal{L}(\phi)} + \overline{\lambda} \cdot \overline{\sigma} \cdot \overline{\phi} = 0,$$
$$\overline{\beta_1} \cdot \overline{\phi(a)} + \overline{\beta_2} \cdot \overline{\phi'(a)} = \overline{\beta_3} \cdot \overline{\phi(b)} + \overline{\beta_4} \cdot \overline{\phi'(b)} = 0.$$

If z is real then $\bar{z} = z$.

$$\overline{\mathcal{L}(\phi)} + \overline{\lambda}\sigma\overline{\phi} = 0,$$
$$\beta_1 \cdot \overline{\phi(a)} + \beta_2 \cdot \overline{\phi'(a)} = \beta_3 \cdot \overline{\phi(b)} + \beta_4 \cdot \overline{\phi'(b)} = 0.$$

Let $\overline{\phi}$ denote the complex conjugate function of ϕ, i.e., $\overline{\phi}(x) = \overline{\phi(x)}$ for $a \leq x \leq b$.

We have that $\phi = f + ig$, where f and g are real-valued functions. Then $\overline{\phi} = f - ig$. Note that

$$\overline{\phi}' = (f - ig)' = f' - ig' = \overline{f' + ig'} = \overline{\phi'}.$$

It follows that

$$\overline{\mathcal{L}(\phi)} = \overline{(p\phi')' + q\phi} = \overline{(p\phi')'} + q\overline{\phi}$$
$$= \left(\overline{p\phi'}\right)' + q\overline{\phi} = \left(p\overline{\phi}'\right)' + q\overline{\phi} = \mathcal{L}(\overline{\phi}).$$

$$\mathcal{L}(\overline{\phi}) + \overline{\lambda}\sigma\overline{\phi} = 0,$$
$$\beta_1\overline{\phi}(a) + \beta_2\overline{\phi}'(a) = \beta_3\overline{\phi}(b) + \beta_4\overline{\phi}'(b) = 0.$$

If ϕ is an eigenfunction belonging to an eigenvalue λ, then $\overline{\phi}$ is an eigenfunction belonging to the eigenvalue $\overline{\lambda}$.

Assume that $\overline{\lambda} \neq \lambda$. Then

$$\int_a^b \phi(x)\overline{\phi(x)}\sigma(x)\,dx = 0.$$

But $\displaystyle\int_a^b \phi(x)\overline{\phi(x)}\sigma(x)\,dx = \int_a^b |\phi(x)|^2\sigma(x)\,dx > 0.$

Thus $\overline{\lambda} = \lambda \implies \lambda \in \mathbb{R}$.

Some facts about Euclidean space

Euclidean space $\mathbb{R}^3$.

Let $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{u} = (u_1, u_2, u_3)$ be two vectors.

$\mathbf{v} \cdot \mathbf{u} = v_1 u_1 + v_2 u_2 + v_3 u_3$ is the dot product.

$\mathbf{v}$ and $\mathbf{u}$ are orthogonal if $\mathbf{v} \cdot \mathbf{u} = 0$.

$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.

Vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ form an orthonormal basis.

$$\begin{aligned}\mathbf{v} &= v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 \\ &= (\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2 + (\mathbf{v} \cdot \mathbf{e}_3)\mathbf{e}_3.\end{aligned}$$

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be orthogonal nonzero vectors. They form a basis in $\mathbb{R}^3$ so that for any $\mathbf{u} \in \mathbb{R}^3$ we have

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

Note that $\mathbf{u} \cdot \mathbf{v}_n = c_n\mathbf{v}_n \cdot \mathbf{v}_n$ so that $c_n = \dfrac{\mathbf{u} \cdot \mathbf{v}_n}{\mathbf{v}_n \cdot \mathbf{v}_n}$.

Pythagorean theorem implies that

$$|\mathbf{u}|^2 = |c_1\mathbf{v}_1|^2 + |c_2\mathbf{v}_2|^2 + |c_3\mathbf{v}_3|^2.$$

Observe that $|c_n\mathbf{v}_n|^2 = |c_n|^2\mathbf{v}_n \cdot \mathbf{v}_n$. Hence

$$\boxed{\mathbf{u} \cdot \mathbf{u} = \frac{|\mathbf{u} \cdot \mathbf{v}_1|^2}{\mathbf{v}_1 \cdot \mathbf{v}_1} + \frac{|\mathbf{u} \cdot \mathbf{v}_2|^2}{\mathbf{v}_2 \cdot \mathbf{v}_2} + \frac{|\mathbf{u} \cdot \mathbf{v}_3|^2}{\mathbf{v}_3 \cdot \mathbf{v}_3}}$$

(Parseval's equality)

Let $\mathbf{v}_1, \mathbf{v}_2$ be orthogonal nonzero vectors.
Given a vector $\mathbf{u} \in \mathbb{R}^3$, let

$$\mathbf{u}_0 = \mathbf{u} - (c_1\mathbf{v}_1 + c_2\mathbf{v}_2), \quad \text{where} \quad c_n = \frac{\mathbf{u} \cdot \mathbf{v}_n}{\mathbf{v}_n \cdot \mathbf{v}_n}.$$

It is easy to check that $\mathbf{u}_0 \cdot \mathbf{v}_1 = \mathbf{u}_0 \cdot \mathbf{v}_2 = 0$ so that $\mathbf{u}_0 \cdot (\mathbf{u} - \mathbf{u}_0) = 0$.

Pythagorean theorem implies that

$$|\mathbf{u}|^2 = |c_1\mathbf{v}_1|^2 + |c_2\mathbf{v}_2|^2 + |\mathbf{u}_0|^2 \geq |c_1\mathbf{v}_1|^2 + |c_2\mathbf{v}_2|^2.$$

Since $|c_n\mathbf{v}_n|^2 = |c_n|^2\mathbf{v}_n \cdot \mathbf{v}_n$, we get

$$\boxed{\mathbf{u} \cdot \mathbf{u} \geq \frac{|\mathbf{u} \cdot \mathbf{v}_1|^2}{\mathbf{v}_1 \cdot \mathbf{v}_1} + \frac{|\mathbf{u} \cdot \mathbf{v}_2|^2}{\mathbf{v}_2 \cdot \mathbf{v}_2}}$$

(Bessel's inequality)

Suppose A and B are linear operators in $\mathbb{R}^3$.
We say that B is **adjoint** to A (denoted $B = A^*$) if

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot B\mathbf{v} \quad \text{for all} \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^3.$$

Let $A = (a_{ij})_{1 \le i,j \le 3}$, $B = (b_{ij})_{1 \le i,j \le 3}$.

Then $A\mathbf{e}_j = a_{1j}\mathbf{e}_1 + a_{2j}\mathbf{e}_2 + a_{3j}\mathbf{e}_3$, hence
$a_{ij} = A\mathbf{e}_j \cdot \mathbf{e}_i$. Similarly, $b_{ij} = B\mathbf{e}_j \cdot \mathbf{e}_i = \mathbf{e}_i \cdot B\mathbf{e}_j$.

It follows that $a_{ij} = b_{ji}$, i.e., B is the transpose of A.

A is called **self-adjoint** if $A = A^*$.
Self-adjoint operators have only real eigenvalues.

Suppose $\mathbf{v}_1$, $\mathbf{v}_2$ are eigenvectors of A belonging to eigenvalues λ_1, λ_2. Then

$$\lambda_1\mathbf{v}_1 \cdot \mathbf{v}_2 = A\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot A\mathbf{v}_2 = \lambda_2\mathbf{v}_1 \cdot \mathbf{v}_2.$$

If $\lambda_1 \neq \lambda_2$ then $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

From Euclidean space to Hilbert space

Hilbert space is an infinite-dimensional analogue of Euclidean space. One realization is

$$L_2[a,b] = \{f : \int_a^b |f(x)|^2\,dx < \infty\}.$$

Inner product of functions:

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx.$$

Since $|fg| \leq \frac{1}{2}(|f|^2 + |g|^2)$, the inner product is well defined for any $f, g \in L_2[a,b]$.

Norm of a function: $\|f\| = \sqrt{\langle f, f\rangle}$.

Convergence: we say that $f_n \to f$ **in the mean** if $\|f - f_n\| \to 0$ as $n \to \infty$.

Functions $f, g \in L_2[a, b]$ are called **orthogonal** if $\langle f, g \rangle = 0$.

Alternative inner product:

$$\langle f, g \rangle_w = \int_a^b f(x)g(x)w(x)\,dx,$$

where w is the **weight** function.

Functions f and g are called **orthogonal with weight** w if $\langle f, g \rangle_w = 0$.

A set $f_1, f_2, \dots$ of pairwise orthogonal nonzero functions is called **complete** if it is maximal, i.e., there is no nonzero function g such that $\langle g, f_n \rangle = 0$, $n = 1, 2, \dots$.

A complete set forms a **basis** of the Hilbert space, that is, each function $g \in L_2[a, b]$ can be expanded into a series

$$g = \sum_{n=1}^{\infty} c_n f_n$$

that converges in the mean.

The expansion is unique: $c_n = \dfrac{\langle g, f_n \rangle}{\langle f_n, f_n \rangle}$.