Math 412-501

Theory of Partial Differential Equations

Lecture 3-1: Heat equation in an arbitrary domain. Spectrum of Laplace's operator.

Heat conduction in an arbitrary domain

Initial-boundary value problem:

$$\frac{\partial u}{\partial t} = k\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big), \quad (x, y) \in D,$$

$u(x, y, 0) = f(x, y), \quad (x, y) \in D,$

Boundary condition: $\quad u\big|_{\partial D} = 0,$

i.e., $u(x, y, t) = 0$ for $(x, y) \in \partial D$.

(Dirichlet condition)

Alternative boundary condition: $\quad \dfrac{\partial u}{\partial n}\Big|_{\partial D} = 0,$

where $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$ is the **normal derivative**.

(Neumann condition)

Mixed boundary condition:

$\partial D = \gamma_1 \sqcup \gamma_2$ (disjoint union),

$$u\big|_{\gamma_1} = 0, \quad \frac{\partial u}{\partial n}\Big|_{\gamma_2} = 0.$$

Boundary condition of the **third kind**:

$$\left(\frac{\partial u}{\partial n} + \alpha u\right)\Big|_{\partial D} = 0, \text{ where } \alpha \text{ is a function on } \partial D.$$

We search for the solution $u(x, y, t)$ as a superposition of solutions with separated variables that satisfy the boundary conditions.

For a general domain, we can only separate the time variable from the others.

Separation of variables: $u(x, y, t) = \phi(x, y)G(t)$.
Substitute this into the heat equation:

$$\phi(x, y)\frac{dG}{dt} = k\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)G(t).$$

Divide both sides by $k \cdot \phi(x, y)G(t) = k \cdot u(x, y, t)$:

$$\frac{1}{kG} \cdot \frac{dG}{dt} = \frac{1}{\phi} \cdot \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right).$$

It follows that

$$\frac{1}{kG} \cdot \frac{dG}{dt} = \frac{1}{\phi} \cdot \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = -\lambda,$$

where λ is a separation constant.

The time variable has been separated:

$$\frac{dG}{dt} = -\lambda k G, \qquad \nabla^2 \phi = -\lambda \phi.$$

Proposition Suppose G and ϕ are solutions of the above differential equations for the same value of λ. Then $u(x, y, t) = \phi(x, y)G(t)$ is a solution of the heat equation.

Boundary condition $u|_{\partial D} = 0$ holds if $\phi|_{\partial D} = 0$.

Boundary condition $\frac{\partial u}{\partial n}\Big|_{\partial D} = 0$ holds if $\frac{\partial \phi}{\partial n}\Big|_{\partial D} = 0$.

Eigenvalue problem:

$$\nabla^2\phi = -\lambda\phi, \quad \phi|_{\partial D} = 0.$$

(Dirichlet Laplacian)

Alternative eigenvalue problem:

$$\nabla^2\phi = -\lambda\phi, \quad \frac{\partial\phi}{\partial n}\bigg|_{\partial D} = 0.$$

(Neumann Laplacian)

We assume that there are **eigenvalues** $\lambda_1, \lambda_2, \ldots$ and corresponding **eigenfunctions** $\phi_1(x,y), \phi_2(x,y), \ldots$.

Dependence on t:

$$G'(t) = -\lambda k G(t) \implies G(t) = C_0 e^{-\lambda k t}$$

Solution of the boundary value problem:

$$u(x, y, t) = e^{-\lambda_n k t}\phi_n(x, y).$$

We are looking for the solution of the initial-boundary value problem as a superposition of solutions with separated variables.

$$u(x, y, t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n k t}\phi_n(x, y)$$

How do we find coefficients c_n?
Substitute the series into the initial condition $u(x, y, 0) = f(x, y)$.

$$f(x, y) = \sum_{n=1}^{\infty} c_n \phi_n(x, y)$$

How do we solve the heat conduction problem?

$$\frac{\partial u}{\partial t} = k\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big), \quad (x,y) \in D,$$
$$u(x,y,0) = f(x,y), \quad (x,y) \in D,$$
$$u\big|_{\partial D} = 0.$$

- Expand f into eigenfunctions of the Dirichlet Laplacian:

$$f(x,y) = \sum_{n=1}^{\infty} c_n \phi_n(x,y).$$

- Write the solution:

$$u(x,y,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n k t} \phi_n(x,y).$$

Spectrum of the Laplacian

Eigenvalue problem:

$$\nabla^2\phi + \lambda\phi = 0 \quad \text{in} \quad D,$$
$$\left(\alpha\phi + \beta\frac{\partial\phi}{\partial n}\right)\Big|_{\partial D} = 0,$$

where α, β are piecewise continuous functions on ∂D such that $|\alpha| + |\beta| \neq 0$ everywhere on ∂D.

We assume that ∂D is piecewise smooth.

The PDE is called the **Helmholtz equation**.

Boundary condition covers all cases considered.

The eigenvalue problem is the many-dimensional analog of the Sturm-Liouville eigenvalue problem.

The eigenvalues of the problem are eigenvalues of the negative Laplacian $-\nabla^2$.

The set of eigenvalues of an operator is called its **spectrum**. Properties of eigenvalues and eigenfunctions are called **spectral properties**.

The Laplacian has **six** important spectral properties.

Property 1. All eigenvalues are real.

Property 2. All eigenvalues can be arranged in the ascending order

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots$$

so that $\lambda_n \to \infty$ as $n \to \infty$.

This means that:

- there are infinitely many eigenvalues;
- there is a smallest eigenvalue;
- on any finite interval, there are only finitely many eigenvalues.

Remark. For the Dirichlet Laplacian, $\lambda_1 > 0$. For the Neumann Laplacian, $\lambda_1 = 0$.

The set of eigenfunctions corresponding to a particular eigenvalue λ together with zero function form a linear space. The dimension of this space is called the **multiplicity** of λ.

An eigenvalue is **simple** if it is of multiplicity 1. Then the eigenfunction is unique up to multiplication by a scalar.
Otherwise the eigenvalue is called **multiple**.

Property 3. An eigenvalue λ_n may be multiple but its multiplicity is finite.

Moreover, the smallest eigenvalue λ_1 is simple, and the corresponding eigenfunction ϕ_1 has no zeros inside the domain D.

Property 4. Eigenfunctions corresponding to different eigenvalues are **orthogonal** relative to the inner product

$$\langle f, g \rangle = \iint_D f(x, y)\overline{g(x, y)}\, dx\, dy.$$

That is, $\langle \phi, \psi \rangle = 0$ whenever ϕ and ψ are eigenfunctions corresponding to different eigenvalues.

Property 5. Any eigenfunction ϕ can be related to its eigenvalue λ through the **Rayleigh quotient**:

$$\lambda = \frac{-\oint_{\partial D} \phi \frac{\partial \phi}{\partial n} \, ds + \iint_D |\nabla \phi|^2 \, dx \, dy}{\iint_D |\phi|^2 \, dx \, dy}.$$

Property 6. There exists a sequence $\phi_1, \phi_2, \ldots$ of pairwise orthogonal eigenfunctions that is **complete** in the Hilbert space $L_2(D)$.

Any square-integrable function $f \in L_2(D)$ is expanded into a series

$$f(x, y) = \sum_{n=1}^{\infty} c_n \phi_n(x, y),$$

that converges in the mean. The series is unique:

$$c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

If f is piecewise smooth then the series converges pointwise to f at points of continuity.

Example.

$$\nabla^2\phi = -\lambda\phi \quad \text{in} \quad D = \{(x,y) \mid 0 < x < L,\ 0 < y < H\},$$

$$\phi(0,y) = \phi(L,y) = 0, \qquad \phi(x,0) = \phi(x,H) = 0.$$

This problem can be solved by separation of variables.

Eigenfunctions $\phi_{nm}(x,y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}$, $n,m \geq 1$.

Corresponding eigenvalues: $\lambda_{nm} = (\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2$.

Thus the double Fourier sine series is the expansion in eigenfunctions of the Dirichlet Laplacian in a rectangle.

Similarly, the double Fourier cosine series is the expansion in eigenfunctions of the Neumann Laplacian in a rectangle.