Math 412-501

Theory of Partial Differential Equations

Lecture 3-2: Spectral properties of the Laplacian. Bessel functions.

Eigenvalue problem:

$$\nabla^2\phi + \lambda\phi = 0 \quad \text{in} \quad D,$$
$$\left(\alpha\phi + \beta\frac{\partial\phi}{\partial \boldsymbol{n}}\right)\Big|_{\partial D} = 0,$$

where α, β are piecewise continuous real functions on ∂D such that $|\alpha| + |\beta| \neq 0$ everywhere on ∂D.

We assume that the boundary ∂D is piecewise smooth.

6 spectral properties of the Laplacian

Property 1. All eigenvalues are real.

Property 2. All eigenvalues can be arranged in the ascending order

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots$$

so that $\lambda_n \to \infty$ as $n \to \infty$.

This means that:

- there are infinitely many eigenvalues;
- there is a smallest eigenvalue;
- on any finite interval, there are only finitely many eigenvalues.

Property 3. An eigenvalue λ_n may be multiple but its multiplicity is finite.

Moreover, the smallest eigenvalue λ_1 is simple, and the corresponding eigenfunction ϕ_1 has no zeros inside the domain D.

Property 4. Eigenfunctions corresponding to different eigenvalues are **orthogonal** relative to the inner product

$$\langle f, g \rangle = \iint_D f(x, y)\overline{g(x, y)}\, dx\, dy.$$

Property 5. Any eigenfunction ϕ can be related to its eigenvalue λ through the **Rayleigh quotient**:

$$\lambda = \frac{-\oint_{\partial D} \phi \frac{\partial \phi}{\partial n} \, ds + \iint_D |\nabla \phi|^2 \, dx \, dy}{\iint_D |\phi|^2 \, dx \, dy}.$$

Property 6. There exists a sequence $\phi_1, \phi_2, \ldots$ of pairwise orthogonal eigenfunctions that is **complete** in the Hilbert space $L_2(D)$.

Any square-integrable function $f \in L_2(D)$ is expanded into a series

$$f(x, y) = \sum_{n=1}^{\infty} c_n \phi_n(x, y),$$

that converges in the mean. The series is unique:

$$c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

If f is piecewise smooth then the series converges pointwise to f at points of continuity.

Rayleigh quotient

Suppose that $\nabla^2\phi = -\lambda\phi$ in the domain D.

Multiply both sides by ϕ and integrate over D:

$$\iint_D \phi\,\nabla^2\phi\,dx\,dy = -\lambda \iint_D |\phi|^2\,dx\,dy.$$

Green's formula:

$$\boxed{\iint_D \psi\,\nabla^2\phi\,dA = \oint_{\partial D} \psi\,\frac{\partial\phi}{\partial n}\,ds - \iint_D \nabla\psi\cdot\nabla\phi\,dA}$$

This is an analog of integration by parts. Now

$$\oint_{\partial D} \phi\,\frac{\partial\phi}{\partial n}\,ds - \iint_D |\nabla\phi|^2\,dx\,dy = -\lambda \iint_D |\phi|^2\,dx\,dy.$$

It follows that

$$\lambda = \frac{-\oint_{\partial D} \phi \frac{\partial \phi}{\partial n}\, ds + \iint_D |\nabla \phi|^2 \, dx\, dy}{\iint_D |\phi|^2 \, dx\, dy}.$$

If ϕ satisfies the boundary condition $\phi|_{\partial D} = 0$ or $\left.\frac{\partial \phi}{\partial n}\right|_{\partial D} = 0$ (or mixed), then the one-dimensional integral vanishes. In particular, $\lambda \geq 0$.

If $\frac{\partial \phi}{\partial n} + \alpha\phi = 0$ on ∂D, then

$$-\oint_{\partial D} \phi \frac{\partial \phi}{\partial n}\, ds = \oint_{\partial D} \alpha |\phi|^2 \, ds.$$

In particular, if $\alpha \geq 0$ everywhere on ∂D, then $\lambda \geq 0$.

Self-adjointness

$$\iint_D \psi\,\nabla^2\phi\,dx\,dy = \oint_{\partial D} \psi\,\frac{\partial\phi}{\partial n}\,ds - \iint_D \nabla\psi\cdot\nabla\phi\,dx\,dy$$

(Green's first identity)

$$\iint_D (\phi\,\nabla^2\psi - \psi\,\nabla^2\phi)\,dx\,dy = \oint_{\partial D}\left(\phi\,\frac{\partial\psi}{\partial n} - \psi\,\frac{\partial\phi}{\partial n}\right)ds$$

(Green's second identity)

If ϕ and ψ satisfy the same boundary condition

$$\left(\alpha\phi + \beta\frac{\partial\phi}{\partial n}\right)\Big|_{\partial D} = \left(\alpha\psi + \beta\frac{\partial\psi}{\partial n}\right)\Big|_{\partial D} = 0$$

then $\phi\,\frac{\partial\psi}{\partial n} - \psi\,\frac{\partial\phi}{\partial n} = 0$ everywhere on ∂D.

If ϕ and ψ satisfy the same boundary condition then

$$\iint_D (\phi\, \nabla^2 \psi - \psi\, \nabla^2 \phi)\, dx\, dy = 0.$$

If ϕ and ψ are complex-valued functions then also

$$\iint_D (\phi\, \overline{\nabla^2 \psi} - \overline{\psi}\, \nabla^2 \phi)\, dx\, dy = 0$$

(because $\overline{\nabla^2 \psi} = \nabla^2 \overline{\psi}$ and $\overline{\psi}$ satisfies the same boundary condition as ψ).

Thus $\langle \nabla^2 \phi, \psi \rangle = \langle \phi, \nabla^2 \psi \rangle$, where

$$\langle f, g \rangle = \iint_D f(x,y) \overline{g(x,y)}\, dx\, dy.$$

Eigenvalue problem:

$$\nabla^2\phi + \lambda\phi = 0 \quad \text{in} \quad D,$$

$$\left(\alpha\phi + \beta\frac{\partial\phi}{\partial \boldsymbol{n}}\right)\Big|_{\partial D} = 0.$$

The Laplacian ∇^2 is self-adjoint in the subspace of functions satisfying the boundary condition.

Suppose ϕ is an eigenfunction belonging to an eigenvalue λ. Let us show that $\lambda \in \mathbb{R}$.

Since $\nabla^2\phi = -\lambda\phi$, we have that

$$\langle\nabla^2\phi, \phi\rangle = \langle-\lambda\phi, \phi\rangle = -\lambda\langle\phi, \phi\rangle,$$

$$\langle\phi, \nabla^2\phi\rangle = \langle\phi, -\lambda\phi\rangle = -\bar{\lambda}\langle\phi, \phi\rangle.$$

Now $\langle\nabla^2\phi, \phi\rangle = \langle\phi, \nabla^2\phi\rangle$ and $\langle\phi, \phi\rangle > 0$ imply $\lambda \in \mathbb{R}$.

Suppose ϕ_1 and ϕ_2 are eigenfunctions belonging to different eigenvalues λ_1 and λ_2.

Let us show that $\langle \phi_1, \phi_2 \rangle = 0$.

Since $\nabla^2 \phi_1 = -\lambda_1 \phi_1$, $\nabla^2 \phi_2 = -\lambda_2 \phi_2$, we have that

$$\langle \nabla^2 \phi_1, \phi_2 \rangle = \langle -\lambda_1 \phi_1, \phi_2 \rangle = -\lambda_1 \langle \phi_1, \phi_2 \rangle,$$
$$\langle \phi_1, \nabla^2 \phi_2 \rangle = \langle \phi_1, -\lambda_2 \phi_2 \rangle = -\overline{\lambda}_2 \langle \phi_1, \phi_2 \rangle.$$

But $\langle \nabla^2 \phi_1, \phi_2 \rangle = \langle \phi_1, \nabla^2 \phi_2 \rangle$, hence

$$-\lambda_1 \langle \phi_1, \phi_2 \rangle = -\overline{\lambda}_2 \langle \phi_1, \phi_2 \rangle.$$

We already know that $\overline{\lambda}_2 = \lambda_2$. Also, $\lambda_1 \neq \lambda_2$.
It follows that $\langle \phi_1, \phi_2 \rangle = 0$.

The main purpose of the Rayleigh quotient

Consider a functional (function on functions)

$$RQ[\phi] = \frac{-\oint_{\partial D} \phi \frac{\partial \phi}{\partial n} \, ds + \iint_D |\nabla \phi|^2 \, dx \, dy}{\iint_D |\phi|^2 \, dx \, dy}.$$

If ϕ is an eigenfunction of $-\nabla^2$ in the domain D with some boundary condition, then $RQ[\phi]$ is the corresponding eigenvalue.

What if ϕ is not?

Let $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n \leq \lambda_{n+1} \leq \ldots$ be eigenvalues of a particular eigenvalue problem **counted with multiplicities**.

That is, a simple eigenvalue appears once in this sequence, an eigenvalue of multiplicity two appears twice, and so on.

There is a complete orthogonal system $\phi_1, \phi_2, \ldots$ in the Hilbert space $L_2(D)$ such that ϕ_n is an eigenfunction belonging to λ_n.

Theorem (i) $\lambda_1 = \min RQ[\phi]$, where the minimum is taken over all nonzero functions ϕ which are differentiable in D and satisfy the boundary condition. Moreover, if $RQ[\phi] = \lambda_1$ then ϕ is an eigenfunction.

(ii) $\lambda_n = \min RQ[\phi]$, where the minimum is taken over all nonzero functions ϕ which are differentiable in D, satisfy the boundary condition, and such that $\langle \phi, \phi_k \rangle = 0$ for $1 \leq k < n$. Moreover, the minimum is attained only on eigenfunctions.

Main idea of the proof: $RQ[\phi] = \dfrac{\langle -\nabla^2 \phi, \phi \rangle}{\langle \phi, \phi \rangle}$.

(see Haberman 5.6)

Spectral properties of the Laplacian in a circle

Eigenvalue problem:

$$\nabla^2\phi + \lambda\phi = 0 \quad \text{in} \quad D = \{(x, y) : x^2 + y^2 \le R^2\},$$

$$u\big|_{\partial D} = 0.$$

In polar coordinates (r, θ):

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \lambda\phi = 0$$

$$(0 < r < R, \ -\pi < \theta < \pi),$$

$$\phi(R, \theta) = 0 \qquad (-\pi < \theta < \pi).$$

Additional boundary conditions:

$$|\phi(0,\theta)| < \infty \qquad (-\pi < \theta < \pi),$$

$$\phi(r,-\pi) = \phi(r,\pi), \;\; \frac{\partial\phi}{\partial\theta}(r,-\pi) = \frac{\partial\phi}{\partial\theta}(r,\pi) \qquad (0 < r < R).$$

Separation of variables: $\phi(r,\theta) = f(r)h(\theta)$.
Substitute this into the equation:

$$f''(r)h(\theta) + r^{-1}f'(r)h(\theta) + r^{-2}f(r)h''(\theta) + \lambda f(r)h(\theta) = 0.$$

Divide by $f(r)h(\theta)$ and multiply by r^2:

$$\frac{r^2 f''(r) + r\,f'(r) + \lambda r^2 f(r)}{f(r)} + \frac{h''(\theta)}{h(\theta)} = 0.$$

It follows that

$$\frac{r^2 f''(r) + r\, f'(r) + \lambda r^2 f(r)}{f(r)} = -\frac{h''(\theta)}{h(\theta)} = \mu = \text{const.}$$

The variables have been separated:

$$r^2 f'' + r f' + (\lambda r^2 - \mu) f = 0,$$
$$h'' = -\mu h.$$

Boundary conditions $\phi(R, \theta) = 0$ and $|\phi(0, \theta)| < \infty$ hold if $f(R) = 0$ and $|f(0)| < \infty$.

Boundary conditions $\phi(r, -\pi) = \phi(r, \pi)$ and $\frac{\partial \phi}{\partial \theta}(r, -\pi) = \frac{\partial \phi}{\partial \theta}(r, \pi)$ hold if $h(-\pi) = h(\pi)$ and $h'(-\pi) = h'(\pi)$.

Eigenvalue problem:

$$h'' = -\mu h, \quad h(-\pi) = h(\pi),\ h'(-\pi) = h'(\pi).$$

Eigenvalues: $\mu_m = m^2,\ m = 0, 1, 2, \ldots$.
$\mu_0 = 0$ is simple, the others are of multiplicity 2.

Eigenfunctions: $h_0 = 1,\ h_m(\theta) = \cos m\theta$ and $\tilde{h}_m(\theta) = \sin m\theta$ for $m \geq 1$.

Dependence on r:

$$r^2 f'' + rf' + (\lambda r^2 - \mu)f = 0, \qquad f(R) = 0,\ |f(0)| < \infty.$$

We may assume that $\mu = m^2$, $m = 0, 1, 2, \ldots$.
Also, we know that $\lambda > 0$ (Rayleigh quotient!).

New variable $z = \sqrt{\lambda} \cdot r$ removes dependence on λ:

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - m^2)f = 0.$$

This is **Bessel's differential equation** of order m.
Solutions are called **Bessel functions** of order m.