Math 412-501

Theory of Partial Differential Equations

Lecture 3-3: Bessel functions.

Spectral properties of the Laplacian in a circle

Eigenvalue problem:

$$\nabla^2\phi + \lambda\phi = 0 \quad \text{in} \quad D = \{(x,y) : x^2 + y^2 \le R^2\},$$

$$u\big|_{\partial D} = 0.$$

In polar coordinates (r, θ):

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \lambda\phi = 0$$

$$(0 < r < R, \ -\pi < \theta < \pi),$$

$$\phi(R,\theta) = 0 \qquad (-\pi < \theta < \pi).$$

Additional boundary conditions:

$$|\phi(0,\theta)| < \infty \quad (-\pi < \theta < \pi),$$

$$\phi(r,-\pi) = \phi(r,\pi), \ \ \frac{\partial\phi}{\partial\theta}(r,-\pi) = \frac{\partial\phi}{\partial\theta}(r,\pi) \quad (0 < r < R).$$

Separation of variables: $\phi(r,\theta) = f(r)h(\theta)$.
Substitute this into the equation:

$$f''(r)h(\theta) + r^{-1}f'(r)h(\theta) + r^{-2}f(r)h''(\theta) + \lambda f(r)h(\theta) = 0.$$

Divide by $f(r)h(\theta)$ and multiply by r^2:

$$\frac{r^2 f''(r) + r\,f'(r) + \lambda r^2 f(r)}{f(r)} + \frac{h''(\theta)}{h(\theta)} = 0.$$

It follows that

$$\frac{r^2 f''(r) + r\,f'(r) + \lambda r^2 f(r)}{f(r)} = -\frac{h''(\theta)}{h(\theta)} = \mu = \text{const.}$$

The variables have been separated:

$$r^2 f'' + rf' + (\lambda r^2 - \mu) f = 0,$$

$$h'' = -\mu h.$$

Boundary conditions $\phi(R,\theta) = 0$ and $|\phi(0,\theta)| < \infty$ hold if $f(R) = 0$ and $|f(0)| < \infty$.

Boundary conditions $\phi(r,-\pi) = \phi(r,\pi)$ and $\frac{\partial\phi}{\partial\theta}(r,-\pi) = \frac{\partial\phi}{\partial\theta}(r,\pi)$ hold if $h(-\pi) = h(\pi)$ and $h'(-\pi) = h'(\pi)$.

Eigenvalue problem:

$$h'' = -\mu h, \quad h(-\pi) = h(\pi),\ h'(-\pi) = h'(\pi).$$

Eigenvalues: $\mu_m = m^2,\ m = 0, 1, 2, \ldots$.
$\mu_0 = 0$ is simple, the others are of multiplicity 2.

Eigenfunctions: $h_0 = 1,\ h_m(\theta) = \cos m\theta$ and $\tilde{h}_m(\theta) = \sin m\theta$ for $m \geq 1$.

Dependence on r:

$$r^2 f'' + rf' + (\lambda r^2 - \mu)f = 0, \qquad f(R) = 0,\ |f(0)| < \infty.$$

We may assume that $\mu = m^2$, $m = 0, 1, 2, \ldots$.
Also, we know that $\lambda > 0$ (Rayleigh quotient!).

New variable $z = \sqrt{\lambda} \cdot r$ removes dependence on λ:

$$\frac{df}{dr} = \sqrt{\lambda}\,\frac{df}{dz}, \qquad \frac{d^2 f}{dr^2} = \lambda\,\frac{d^2 f}{dz^2}.$$

$$\boxed{z^2\,\frac{d^2 f}{dz^2} + z\,\frac{df}{dz} + (z^2 - m^2)f = 0}$$

This is **Bessel's differential equation** of order m.
Solutions are called **Bessel functions** of order m.

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - m^2) f = 0$$

Solutions are well behaved in the interval $(0, \infty)$.

Let f_1 and f_2 be linearly independent solutions. Then the general solution is $f = c_1 f_1 + c_2 f_2$, where c_1, c_2 are constants.

We need to determine the behavior of solutions as $z \to 0$ and as $z \to \infty$.

In a neighborhood of 0, Bessel's equation is a small perturbation of the equidimensional equation

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} - m^2 f = 0.$$

Equidimensional equation:

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} - m^2 f = 0.$$

For $m > 0$, the general solution is $f(z) = c_1 z^m + c_2 z^{-m}$, where c_1, c_2 are constants.

For $m = 0$, the general solution is $f(z) = c_1 + c_2 \log z$, where c_1, c_2 are constants.

We hope that Bessel functions are close to solutions of the equidimensional equation as $z \to 0$.

Theorem For any $m > 0$ there exist Bessel functions f_1 and f_2 of order m such that

$$f_1(z) \sim z^m \quad \text{and} \quad f_2(z) \sim z^{-m} \quad \text{as} \quad z \to 0.$$

Also, there exist Bessel functions f_1 and f_2 of order 0 such that

$$f_1(z) \sim 1 \quad \text{and} \quad f_2(z) \sim \log z \quad \text{as} \quad z \to 0.$$

Remarks. (i) f_1 and f_2 are linearly independent.
(ii) f_1 is determined uniquely while f_2 is not.

$J_m(z)$: **Bessel function of the first kind**,
$Y_m(z)$: **Bessel function of the second kind**.

$J_m(z)$ and $Y_m(z)$ are certain linearly independent Bessel functions of order m.

$J_m(z)$ is regular while $Y_m(z)$ has singularity at 0.

$J_m(z)$ and $Y_m(z)$ are **special functions**.

As $z \to 0$, we have for $m > 0$

$$J_m(z) \sim \frac{1}{2^m\, m!} z^m, \quad Y_m(z) \sim -\frac{2^m (m-1)!}{\pi} z^{-m}.$$

Also, $J_0(z) \sim 1, \quad Y_0(z) \sim \dfrac{2}{\pi} \log z$.

$J_m(z)$ is uniquely determined by its asymptotics as $z \to 0$. Original definition by Bessel:

$$J_m(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \tau - m\tau)\, d\tau.$$

Behavior of the Bessel functions as $z \to \infty$ does not depend on the order m. Any Bessel function f satisfy

$$f(z) = Az^{-1/2} \cos(z - B) + O(z^{-1}) \quad \text{as} \quad z \to \infty,$$

where A, B are constants.

The function f is uniquely determined by A, B, and its order m.

As $z \to \infty$, we have

$$J_m(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4} - \frac{m\pi}{2}\right) + O(z^{-1}),$$

$$Y_m(z) = \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4} - \frac{m\pi}{2}\right) + O(z^{-1}).$$

Let $0 < j_{m,1} < j_{m,2} < \dots$ be zeros of $J_m(z)$ and $0 < y_{m,1} < y_{m,2} < \dots$ be zeros of $Y_m(z)$.
Then the zeros are interlaced:

$$m < y_{m,1} < j_{m,1} < y_{m,2} < j_{m,2} < \dots .$$

Asymptotics of the nth zeros as $n \to \infty$:

$$j_{m,n} \sim (n + \tfrac{1}{2}m - \tfrac{1}{4})\pi, \quad y_{m,n} \sim (n + \tfrac{1}{2}m - \tfrac{3}{4})\pi.$$

Eigenvalues of the Laplacian in a circle

Intermediate eigenvalue problem:

$$r^2 f'' + rf' + (\lambda r^2 - m^2)f = 0, \qquad f(R) = 0, \ |f(0)| < \infty.$$

New variable $z = \sqrt{\lambda} \cdot r$ reduced the equation to Bessel's equation of order m. Hence the general solution is $f(r) = c_1 J_m(\sqrt{\lambda}\, r) + c_2 Y_m(\sqrt{\lambda}\, r)$, where c_1, c_2 are constants.

Singular condition $|f(0)| < \infty$ holds if $c_2 = 0$.
Nonzero solution exists if $J_m(\sqrt{\lambda}\, R) = 0$.

Thus there are infinitely many eigenvalues $\lambda_{m,1}, \lambda_{m,2}, \dots,$
where $\sqrt{\lambda_{m,n}}\, R = j_{m,n}$, i.e., $\lambda_{m,n} = (j_{m,n}/R)^2$.

Summary

Eigenvalue problem:

$$\nabla^2\phi + \lambda\phi = 0 \quad \text{in} \quad D = \{(x, y) : x^2 + y^2 \leq R^2\},$$
$$u\big|_{\partial D} = 0.$$

Eigenvalues: $\lambda_{m,n} = (j_{m,n}/R)^2$, where $m = 0, 1, 2, \ldots,\ n = 1, 2, \ldots,$ and $j_{m,n}$ is the nth zero of the Bessel function J_m.

Eigenfunctions: $\phi_{0,n}(r, \theta) = J_0(j_{0,n}\, r/R)$.
For $m \geq 1$, $\phi_{m,n}(r, \theta) = J_m(j_{m,n}\, r/R) \cos m\theta$ and $\tilde{\phi}_{m,n}(r, \theta) = J_m(j_{m,n}\, r/R) \sin m\theta$.