Math 412-501

Theory of Partial Differential Equations

Lecture 4-4:
Green's function for the wave equation.

Green's function for the heat equation

Green's function $G(x, t; x_0, t_0)$ for the heat equation on the infinite interval satisfies

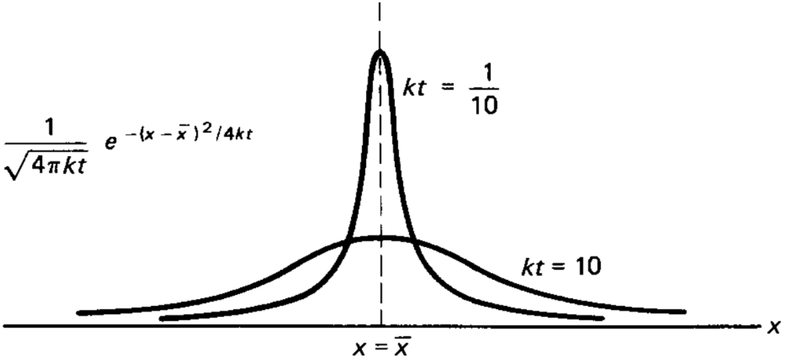
$$\frac{\partial G}{\partial t} = k \frac{\partial^2 G}{\partial x^2} + \delta(x - x_0)\, \delta(t - t_0)$$

subject to the causality principle:

$$G(x, t; x_0, t_0) = 0 \quad \text{for} \quad t < t_0.$$

For $t > t_0$ we have that

$$G(x, t; x_0, t_0) = \frac{1}{\sqrt{4\pi k(t - t_0)}}\, e^{-\frac{(x-x_0)^2}{4k(t-t_0)}}.$$



$\frac{1}{\sqrt{4\pi kt}} e^{-(x-\bar{x})^2/4kt}$
$kt = \frac{1}{10}$
$kt = 10$
x
$x = \bar{x}$

General nonhomogeneous problem

Initial value problem:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad (-\infty < x < \infty,\; t > 0),$$

$$u(x,0) = f(x).$$

Solution: $u(x,t) =$

$$= \int_0^\infty \int_{-\infty}^\infty G(x,t;x_0,t_0)\,Q(x_0,t_0)\,dx_0\,dt_0$$

$$+ \int_{-\infty}^\infty G(x,t;x_0,0)\,f(x_0)\,dx_0.$$

General nonhomogeneous problem

Initial value problem:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad (-\infty < x < \infty,\ t > 0),$$

$$u(x,0) = f(x).$$

Solution: $u(x,t) =$

$$= \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-t_0)}}\, e^{-\frac{(x-x_0)^2}{4k(t-t_0)}}\, Q(x_0,t_0)\, dx_0\, dt_0$$

$$+ \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k t}}\, e^{-\frac{(x-x_0)^2}{4kt}}\, f(x_0)\, dx_0.$$

Green's function for the wave equation

Green's function $G(x, t; x_0, t_0)$ for the infinite interval describes vibrations of an infinite string caused by an instant unit force which is applied at time t_0 to the point x_0.

Formally, G solves the equation

$$\frac{\partial^2 G}{\partial t^2} = c^2 \frac{\partial^2 G}{\partial x^2} + \delta(x - x_0)\,\delta(t - t_0)$$

subject to the condition

$$G(x, t; x_0, t_0) = 0 \quad \text{for} \quad t < t_0.$$

(causality principle)

Apply the Fourier transform (relative to x) to both sides of the equation:

$$\mathcal{F}_x\left[\frac{\partial^2 G}{\partial t^2}\right] = c^2\,\mathcal{F}_x\left[\frac{\partial^2 G}{\partial x^2}\right] + \mathcal{F}_x[\delta(x-x_0)]\,\delta(t-t_0).$$

Let $\widehat{G}(\omega, t; x_0, t_0)$ denote the Fourier transform of G relative to x:

$$\widehat{G}(\omega, t; x_0, t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(x, t; x_0, t_0)e^{-i\omega x}\,dx.$$

$$\mathcal{F}_x\left[\frac{\partial^2 G}{\partial t^2}\right] = \frac{\partial^2 \widehat{G}}{\partial t^2}, \quad \mathcal{F}_x\left[\frac{\partial^2 G}{\partial x^2}\right] = (i\omega)^2\widehat{G} = -\omega^2\widehat{G},$$

$$\mathcal{F}_x[\delta(x-x_0)](\omega) = \frac{1}{2\pi}\,e^{-i\omega x_0}.$$

$$\Longrightarrow \quad \frac{\partial^2 \widehat{G}}{\partial t^2} = -c^2\omega^2 \widehat{G} + \frac{e^{-i\omega x_0}}{2\pi}\,\delta(t - t_0).$$

Besides, $\widehat{G}(\omega, t; x_0, t_0) = 0$ for $t < t_0$.

It follows that

$$\widehat{G}(\omega, t; x_0, t_0) = \begin{cases} 0 & \text{for } t < t_0, \\ ae^{ic\omega t} + be^{-ic\omega t} & \text{for } t > t_0, \end{cases}$$

where $a = a(\omega, x_0, t_0)$, $b = b(\omega, x_0, t_0)$;

$$\frac{\partial \widehat{G}}{\partial t}\Big|_{t=t_0+} - \frac{\partial \widehat{G}}{\partial t}\Big|_{t=t_0-} = \frac{e^{-i\omega x_0}}{2\pi};$$

$$\widehat{G}\Big|_{t=t_0-} = \widehat{G}\Big|_{t=t_0+}.$$

$$\widehat{G}\Big|_{t=t_0+} = ae^{ic\omega t_0} + be^{-ic\omega t_0} = 0,$$

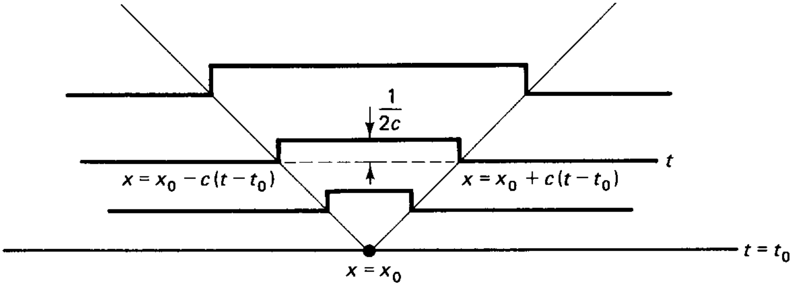
$$\frac{\partial \widehat{G}}{\partial t}\Big|_{t=t_0+} = ic\omega \cdot ae^{ic\omega t_0} - ic\omega \cdot be^{-ic\omega t_0} = \frac{e^{-i\omega x_0}}{2\pi}.$$

Then $\quad a = \dfrac{e^{-i\omega x_0}}{4\pi ic\omega}\, e^{-ic\omega t_0}, \qquad b = -\dfrac{e^{-i\omega x_0}}{4\pi ic\omega}\, e^{ic\omega t_0}.$

Hence $\quad \widehat{G} = \dfrac{e^{-i\omega x_0}}{4\pi ic\omega}\left(e^{ic\omega(t-t_0)} - e^{-ic\omega(t-t_0)}\right)$

$$= \frac{e^{-i\omega x_0}}{2c}\,\frac{\sin(c\omega(t-t_0))}{\pi\omega} \qquad \text{if } \ t > t_0.$$

$$G(x,t;x_0,t_0) = \begin{cases} \frac{1}{2c} & \text{if } \ |x-x_0| < c(t-t_0), \\ 0 & \text{if } \ |x-x_0| > c(t-t_0). \end{cases}$$



$G(x, t; x_0, t_0)$ as a function of x

Nonhomogeneous problems

Initial value problem #1:

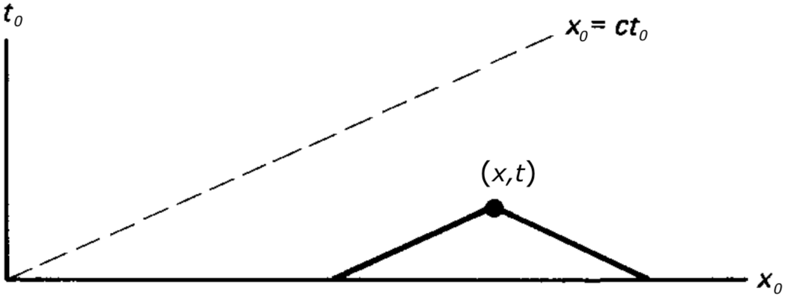
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad (-\infty < x < \infty,\ t > 0),$$

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

Solution: $u(x,t) = \int_0^\infty \int_{-\infty}^\infty G(x,t;x_0,t_0)\, Q(x_0,t_0)\, dx_0\, dt_0$

$$= \frac{1}{2c} \iint_{D_{x,t}} Q(x_0,t_0)\, dx_0\, dt_0,$$

where $D_{x,t} = \{(x_0,t_0) : 0 < t_0 < t - c^{-1}|x - x_0|\}.$



Domain of influence $D_{x,t}$

Nonhomogeneous problems

Initial value problem #2:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty,\ t > 0),$$

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

Solution: $u(x,t) =$

$$= \int_{-\infty}^{\infty} G(x,t;x_0,0)\, g(x_0)\, dx_0$$

$$= \frac{1}{2c} \int_{x-ct}^{x+ct} g(x_0)\, dx_0.$$

Nonhomogeneous problems

Initial value problem #3:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty,\ t > 0),$$

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

Solution: $u(x,t) = \int_{-\infty}^{\infty} G_1(x,t;x_0,0)\, f(x_0)\, dx_0,$

where $G_1(x,t;x_0,t_0)$ is the solution of the equation

$$\frac{\partial^2 G_1}{\partial t^2} = c^2 \frac{\partial^2 G_1}{\partial x^2} + \delta(x-x_0)\,\delta'(t-t_0)$$

subject to the causality principle.

Since $\dfrac{\partial^2 G}{\partial t^2} = c^2 \dfrac{\partial^2 G}{\partial x^2} + \delta(x - x_0)\,\delta(t - t_0)$,

it follows that $G_1 = -\dfrac{\partial G}{\partial t_0}$.

Let H denote the Heaviside function: $H(z) = 0$ for $z < 0$ and $H(z) = 1$ for $z > 0$. Then

$$G(x, t; x_0, t_0) = \frac{1}{2c}\Big(H\big(x - x_0 + c(t - t_0)\big) - \\ - H\big(x - x_0 - c(t - t_0)\big)\Big),$$

$$\frac{\partial G}{\partial t_0}(x, t; x_0, t_0) = -\frac{1}{2}\,\delta\big(x - x_0 + c(t - t_0)\big) - \\ - \frac{1}{2}\,\delta\big(x - x_0 - c(t - t_0)\big).$$

Nonhomogeneous problems

Initial value problem #3:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty,\ t > 0),$$

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

Solution: $u(x,t) =$

$$= -\int_{-\infty}^{\infty} \frac{\partial G}{\partial t_0}(x,t;x_0,0)\, f(x_0)\, dx_0$$

$$= \frac{f(x+ct) + f(x-ct)}{2}.$$

General nonhomogeneous problem

Initial value problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad (-\infty < x < \infty,\ t > 0),$$

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

Solution: $u(x,t) =$

$$= \int_0^\infty \int_{-\infty}^\infty G(x,t;x_0,t_0)\, Q(x_0,t_0)\, dx_0\, dt_0$$

$$- \int_{-\infty}^\infty \frac{\partial G}{\partial t_0}(x,t;x_0,0)\, f(x_0)\, dx_0 + \int_{-\infty}^\infty G(x,t;x_0,0)\, g(x_0)\, dx_0.$$

General nonhomogeneous problem

Initial value problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad (-\infty < x < \infty,\ t > 0),$$

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

Solution: $u(x,t) =$

$$= \frac{1}{2c} \iint_{D_{x,t}} Q(x_0,t_0)\, dx_0\, dt_0$$

$$+ \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(x_0)\, dx_0.$$