Math 412-501

Theory of Partial Differential Equations

Lecture 4-6:
Review for the final exam.

Math 412-501 Fall 2006

Sample problems for the final exam

Any problem may be altered or replaced by a different one!

Some possibly useful information

• Parseval's equality for the complex form of the Fourier series on $(-\pi, \pi)$:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \Longrightarrow \quad \int_{-\pi}^{\pi} |f(x)|^2 \, dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2.$$

• Fourier sine and cosine transforms of the second derivative:

$$S[f''](\omega) = \frac{2}{\pi} f(0)\, \omega - \omega^2 S[f](\omega),$$

$$C[f''](\omega) = -\frac{2}{\pi} f'(0) - \omega^2 C[f](\omega).$$

- Laplace's operator in polar coordinates r, θ:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

- Any nonzero solution of a regular Sturm-Liouville equation

$$(p\phi')' + q\phi + \lambda\sigma\phi = 0 \qquad (a < x < b)$$

satisfies the Rayleigh quotient relation

$$\lambda = \frac{-p\phi\phi'\Big|_a^b + \int_a^b \left(p(\phi')^2 - q\phi^2\right) dx}{\int_a^b \phi^2\sigma\, dx}.$$

- Some table integrals:

$$\int x^2 e^{iax}\,dx = \left(\frac{x^2}{ia} + \frac{2x}{a^2} - \frac{2}{ia^3}\right) e^{iax} + C, \quad a \neq 0;$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} e^{i\beta x}\,dx = \sqrt{\frac{\pi}{\alpha}}\, e^{-\beta^2/(4\alpha)}, \quad \alpha > 0,\ \beta \in \mathbb{R};$$

$$\int_{-\infty}^{\infty} e^{-\alpha|x|} e^{i\beta x}\,dx = \frac{2\alpha}{\alpha^2 + \beta^2}, \quad \alpha > 0,\ \beta \in \mathbb{R}.$$

Problem 1 Let $f(x) = x^2$.

(i) Find the Fourier series (complex form) of $f(x)$ on the interval $(-\pi, \pi)$.

(ii) Rewrite the Fourier series of $f(x)$ in the real form.

(iii) Sketch the function to which the Fourier series converges.

(iv) Use Parseval's equality to evaluate $\sum_{n=1}^{\infty} n^{-4}$.

Problem 1 Let $f(x) = x^2$.

(i) Find the Fourier series (complex form) of $f(x)$ on the interval $(-\pi, \pi)$.

The required series is $\displaystyle\sum_{n=-\infty}^{\infty} c_n e^{inx}$, where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.$$

In particular,

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2\, dx = \frac{1}{2\pi} \frac{x^3}{3} \Big|_{-\pi}^{\pi} = \frac{1}{2\pi} \frac{2\pi^3}{3} = \frac{\pi^2}{3}.$$

If $n \neq 0$ then we have to integrate by parts twice.

To save time, we could instead use the table integral

$$\int x^2 e^{iax}\,dx = \left(\frac{x^2}{ia} + \frac{2x}{a^2} - \frac{2}{ia^3}\right) e^{iax} + C, \quad a \neq 0.$$

According to this integral,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx = \frac{1}{2\pi}\left(-\frac{x^2}{in} + \frac{2x}{n^2} + \frac{2}{in^3}\right) e^{-inx}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\,\frac{2\pi(e^{-in\pi} + e^{in\pi})}{n^2} = \frac{2(-1)^n}{n^2}.$$

Thus

$$x^2 \sim \frac{\pi^2}{3} + \sum_{\substack{-\infty<n<\infty \\ n\neq 0}} \frac{2(-1)^n}{n^2}\,e^{inx}.$$

(ii) Rewrite the Fourier series of $f(x)$ in the real form.

$$\frac{\pi^2}{3} + \sum_{\substack{-\infty<n<\infty \\ n\neq 0}} \frac{2(-1)^n}{n^2} e^{inx} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \left(e^{inx} + e^{-inx}\right)$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx.$$

Thus

$$x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx.$$

(iii) Sketch the function to which the Fourier series converges.

The series converges to the 2π-periodic function that coincides with $f(x)$ for $-\pi \leq x \leq \pi$.

The sum is continuous and piecewise smooth hence the convergence is uniform.

The derivative of the sum has jump discontinuities at points $\pi + 2k\pi$, $k \in \mathbb{Z}$.

The graph is a scalloped curve.

(iv) Use Parseval's equality to evaluate $\sum_{n=1}^{\infty} n^{-4}$.

In our case, Parseval's equality can be written as

$$\langle f, f \rangle = \sum_{n=-\infty}^{\infty} \frac{|\langle f, \phi_n \rangle|^2}{\langle \phi_n, \phi_n \rangle},$$

where $\phi_n(x) = e^{inx}$ and

$$\langle g, h \rangle = \int_{-\pi}^{\pi} g(x)\overline{h(x)}\, dx.$$

Since $c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}$ and $\langle \phi_n, \phi_n \rangle = 2\pi$ for all $n \in \mathbb{Z}$, it can be reduced to an equivalent form

$$\int_{-\pi}^{\pi} |f(x)|^2\, dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Now

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx = \int_{-\pi}^{\pi} x^4 \, dx = \frac{x^5}{5} \Big|_{-\pi}^{\pi} = \frac{2\pi^5}{5},$$

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty} \frac{4}{n^4}.$$

Therefore

$$\frac{1}{2\pi}\,\frac{2\pi^5}{5} = \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty} \frac{4}{n^4}.$$

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{8}\left(\frac{\pi^4}{5} - \frac{\pi^4}{9}\right) = \frac{\pi^4}{90}.$$

Problem 2 Solve Laplace's equation in a disk,

$$\nabla^2 u = 0 \quad (0 \leq r < a), \qquad u(a, \theta) = f(\theta).$$

Laplace's operator in polar coordinates r, θ:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

We search for the solution of the boundary value problem as a superposition of solutions $u(r, \theta) = h(r)\phi(\theta)$ with separated variables.

Solutions with separated variables satisfy periodic boundary conditions

$$u(r, -\pi) = u(r, \pi), \qquad \frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi)$$

and the singular boundary condition

$$|u(0, \theta)| < \infty.$$

Separation of variables provides the following solutions:

$$u_0 = 1, \qquad u_n(r,\theta) = r^n \cos n\theta, \quad \tilde{u}_n(r,\theta) = r^n \sin n\theta, \quad n = 1, 2, \ldots$$

A superposition of these solutions is a series

$$u(r,\theta) = \alpha_0 + \sum\nolimits_{n=1}^{\infty} r^n(\alpha_n \cos n\theta + \beta_n \sin n\theta),$$

where $\alpha_0, \alpha_1, \ldots$ and $\beta_1, \beta_2, \ldots$ are constants. Substituting the series into the boundary condition $u(a,\theta) = f(\theta)$, we get

$$f(\theta) = \alpha_0 + \sum\nolimits_{n=1}^{\infty} a^n(\alpha_n \cos n\theta + \beta_n \sin n\theta).$$

The boundary condition is satisfied if the right-hand side coincides with the Fourier series

$$A_0 + \sum\nolimits_{n=1}^{\infty}(A_n \cos n\theta + B_n \sin n\theta)$$

of the function $f(\theta)$ on $(-\pi, \pi)$.

Hence

$$\alpha_0 = A_0, \qquad \alpha_n = a^{-n} A_n, \quad \beta_n = a^{-n} B_n, \quad n = 1, 2, \ldots$$

and

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (A_n \cos n\theta + B_n \sin n\theta),$$

where

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta, \qquad A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta\, d\theta,$$

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta\, d\theta, \quad n = 1, 2, \ldots$$

Bonus Problem 7 Find a Green function implementing the solution of Problem 2.

The solution of Problem 2:

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (A_n \cos n\theta + B_n \sin n\theta),$$

where

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta_0)\,d\theta_0, \quad A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta_0)\cos n\theta_0\,d\theta_0,$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta_0)\sin n\theta_0\,d\theta_0, \quad n = 1, 2, \dots$$

It can be rewritten as

$$u(r,\theta) = \int_{-\pi}^{\pi} G(r,\theta;\theta_0)\,f(\theta_0)\,d\theta_0,$$

where

$$G(r,\theta;\theta_0) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (\cos n\theta \cos n\theta_0 + \sin n\theta \sin n\theta_0).$$

This is the desired Green function. The expression can be simplified:

$$\begin{aligned} G(r,\theta;\theta_0) &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n (\cos n\theta \cos n\theta_0 + \sin n\theta \sin n\theta_0) \\ &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\theta_0) \\ &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cdot \frac{e^{in(\theta-\theta_0)} + e^{-in(\theta-\theta_0)}}{2} \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\left(ra^{-1}e^{i(\theta-\theta_0)}\right)^n + \frac{1}{2\pi}\sum_{n=1}^{\infty}\left(ra^{-1}e^{-i(\theta-\theta_0)}\right)^n \end{aligned}$$

$$
\begin{aligned}
G(r,\theta;\theta_0) &= \frac{1}{2\pi}\left(\frac{1}{1-ra^{-1}e^{i(\theta-\theta_0)}} + \frac{ra^{-1}e^{-i(\theta-\theta_0)}}{1-ra^{-1}e^{-i(\theta-\theta_0)}}\right) \\
&= \frac{1}{2\pi}\left(\frac{a}{a-re^{i(\theta-\theta_0)}} + \frac{re^{-i(\theta-\theta_0)}}{a-re^{-i(\theta-\theta_0)}}\right) \\
&= \frac{1}{2\pi}\,\frac{a^2-r^2}{(a-re^{i(\theta-\theta_0)})(a-re^{-i(\theta-\theta_0)})} \\
&= \frac{1}{2\pi}\,\frac{a^2-r^2}{a^2-2ar\cos(\theta-\theta_0)+r^2}.
\end{aligned}
$$

(Poisson's formula)

Problem 3 Find Green's function for the boundary value problem

$$\frac{d^2u}{dx^2} - u = f(x) \quad (0 < x < 1), \qquad u'(0) = u'(1) = 0.$$

The Green function $G(x, x_0)$ should satisfy

$$\frac{\partial^2 G}{\partial x^2} - G = \delta(x - x_0), \qquad \frac{\partial G}{\partial x}(0, x_0) = \frac{\partial G}{\partial x}(1, x_0) = 0.$$

It follows that

$$G(x, x_0) = \begin{cases} ae^x + be^{-x} & \text{for } x < x_0, \\ ce^x + de^{-x} & \text{for } x > x_0, \end{cases}$$

where constants a, b, c, d may depend on x_0. Then

$$\frac{\partial G}{\partial x}(x, x_0) = \begin{cases} ae^x - be^{-x} & \text{for } x < x_0, \\ ce^x - de^{-x} & \text{for } x > x_0. \end{cases}$$

The boundary conditions imply that $a = b$ and $ce = de^{-1}$.

Gluing conditions at $x = x_0$ are continuity of the function and jump discontinuity of the first derivative:

$$G(x, x_0) \Big|_{x=x_0-} = G(x, x_0) \Big|_{x=x_0+}, \quad \frac{\partial G}{\partial x} \Big|_{x=x_0+} - \frac{\partial G}{\partial x} \Big|_{x=x_0-} = 1.$$

The two conditions imply that

$$ae^{x_0} + be^{-x_0} = ce^{x_0} + de^{-x_0}, \qquad ce^{x_0} - de^{-x_0} - (ae^{x_0} - be^{-x_0}) = 1.$$

Now we have 4 equations to determine 4 quantities a, b, c, d.

Solution:

$$c = \frac{e^{x_0} + e^{-x_0}}{2(1 - e^2)}, \qquad a = \frac{e^{x_0} + e^{2-x_0}}{2(1 - e^2)},$$

$$d = \frac{e^{x_0} + e^{-x_0}}{2(e^{-2} - 1)}, \qquad b = \frac{e^{x_0} + e^{2-x_0}}{2(1 - e^2)}.$$

Finally,

$$G(x,x_0) = \begin{cases} \dfrac{(e^{x_0}+e^{2-x_0})(e^{x}+e^{-x})}{2(1-e^2)} & \text{for} \quad x < x_0, \\[2ex] \dfrac{(e^{x_0}+e^{-x_0})(e^{x}+e^{2-x})}{2(1-e^2)} & \text{for} \quad x > x_0. \end{cases}$$

Observe that $G(x,x_0) = G(x_0,x)$.

Problem 4 Solve the initial-boundary value problem for the heat equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \pi, \quad t > 0),$$

$$u(x,0) = f(x) \qquad (0 < x < \pi),$$

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(\pi, t) + 2u(\pi, t) = 0.$$

In the process you will discover a sequence of eigenfunctions and eigenvalues, which you should name $\phi_n(x)$ and λ_n.

Describe the λ_n qualitatively (e.g., find an equation for them) but do not expect to find their exact numerical values.

Also, do not bother to evaluate normalization integrals for ϕ_n.

We search for the solution of the initial-boundary value problem as a superposition of solutions $u(x,t) = \phi(x)g(t)$ with separated variables of the heat equation that satisfy the boundary conditions.

We get an equation for g:

$$g' = -\lambda g \implies g(t) = c_0 e^{-\lambda t},$$

and an eigenvalue problem for ϕ:

$$\phi'' = -\lambda\phi, \qquad \phi(0) = 0, \quad \phi'(\pi) + 2\phi(\pi) = 0.$$

This is a regular Sturm-Liouville eigenvalue problem.

Rayleigh quotient:

$$\lambda = \frac{-\phi\phi'\Big|_0^\pi + \displaystyle\int_0^\pi |\phi'(x)|^2\,dx}{\displaystyle\int_0^\pi |\phi(x)|^2\,dx}.$$

Note that $-\phi\phi'\,\Big|_0^\pi = \phi(0)\phi'(0) - \phi(\pi)\phi'(\pi) = 2|\phi(\pi)|^2$.

It follows that $\lambda > 0$.

The eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$ are solutions of the equation

$$-\frac{1}{2}\sqrt{\lambda} = \tan(\sqrt{\lambda}\,\pi),$$

and the corresponding eigenfunctions are $\phi_n(x) = \sin(\sqrt{\lambda_n}\,x)$.

Solutions with separated variables:

$$u_n(x,t) = e^{-\lambda_n t}\phi_n(x) = e^{-\lambda_n t}\sin(\sqrt{\lambda_n}\,x), \quad n = 1, 2, \dots$$

A superposition of these solutions is a series

$$u(x,t) = \sum\nolimits_{n=1}^{\infty} c_n e^{-\lambda_n t}\phi_n(x) = \sum\nolimits_{n=1}^{\infty} c_n e^{-\lambda_n t}\sin(\sqrt{\lambda_n}\,x),$$

where $c_1, c_2, \dots$ are constants. Substituting the series into the initial condition $u(x,0) = f(x)$, we get

$$f(x) = \sum\nolimits_{n=1}^{\infty} c_n\phi_n(x).$$

The initial condition is satisfied if the right-hand side coincides with the generalized Fourier series of the function f, that is, if

$$c_n = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}, \quad n = 1, 2, \dots$$

Problem 5 By the method of your choice, solve the wave equation on the half-line

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \infty,\ -\infty < t < \infty)$$

subject to

$$u(0,t) = 0, \qquad u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

Bonus Problem 6 Solve Problem 5 by a distinctly different method.

Fourier's method: In view of the boundary condition, let us apply the Fourier sine transform with respect to x to both sides of the equation:

$$S\left[\frac{\partial^2 u}{\partial t^2}\right] = S\left[\frac{\partial^2 u}{\partial x^2}\right].$$

Let

$$U(\omega, t) = S[u(\cdot, t)](\omega) = \frac{2}{\pi}\int_0^\infty u(x, t)\sin\omega x\, dx.$$

Then

$$S\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{\partial^2 U}{\partial t^2}, \quad S\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{2}{\pi}u(0, t)\omega - \omega^2 U(\omega, t) = -\omega^2 U(\omega, t).$$

Hence

$$\frac{\partial^2 U}{\partial t^2} = -\omega^2 U(\omega, t).$$

If $\omega \neq 0$ then the general solution of the latter equation is $U(\omega, t) = a \cos \omega t + b \sin \omega t$, where $a = a(\omega)$, $b = b(\omega)$. Applying the Fourier sine transform to the initial conditions, we obtain

$$U(\omega, 0) = F(\omega), \qquad \frac{\partial U}{\partial t}(\omega, 0) = G(\omega),$$

where $F = S[f]$, $G = S[g]$.

It follows that $a(\omega) = F(\omega)$, $b(\omega) = G(\omega)/\omega$.

Now it remains to apply the inverse Fourier sine transform:

$$u(x, t) = \int_0^\infty \left(F(\omega) \cos \omega t + \frac{G(\omega)}{\omega} \sin \omega t \right) \sin \omega x \, d\omega,$$

where

$$F(\omega) = \frac{2}{\pi} \int_0^\infty f(x_0) \sin \omega x_0 \, dx_0, \quad G(\omega) = \frac{2}{\pi} \int_0^\infty g(x_0) \sin \omega x_0 \, dx_0.$$

D'Alembert's method: Define $f(x)$ and $g(x)$ for negative x to be the odd extensions of the functions given for positive x, i.e., $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all $x > 0$.

By d'Alembert's formula, the function

$$u(x,t) = \frac{1}{2}\big(f(x+t) + f(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(x_0)\,dx_0$$

is the solution of the wave equation that satisfies the initial conditions

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

on the entire line.

Since f and g are odd functions, it follows that $u(x,t)$ is also odd as a function of x. As a consequence, $u(0,t) = 0$ for all t.