Sample problems for Exam 1

Any problem may be altered, removed or replaced by a different one!

Problem 1. Consider an operation $*$ defined on the set $\mathbb{Z}$ of integers by $a * b = a + b - 2$. Does this operation provide the integers with a group structure?

Problem 2. Suppose $(S, *)$ is a semigroup satisfying the following two conditions: (i) there exists $e \in S$ such that $e * g = g$ for all $g \in S$ (existence of a left identity element), and (ii) for any $g \in S$ there exists $g' \in S$ such that $g' * g = e$ (existence of a left inverse). Prove that $(S, *)$ is a group.

Problem 3. Prove that the group $(\mathbb{Q} \setminus \{0\}, \cdot)$ is not cyclic.

Problem 4. Let G be a group of order 125. Show that G contains an element of order 5.

Problem 5. Find the order and the sign of the permutation $\sigma = (1\,2)(3\,4\,5\,6)(1\,2\,3\,4)(5\,6)$.

Problem 6. Suppose $\pi, \sigma \in S_5$ are permutations of order 3. What are possible values for the order of the permutation $\pi\sigma$?

Problem 7. Find all subgroups of the alternating group A_4.

Problem 8. Determine which of the following groups of order 12 are isomorphic and which are not: $\mathbb{Z}_{12}$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_6$, $S_3 \times \mathbb{Z}_2$, A_4 and D_6.

Problem 9. Find an example of an abelian group G and its subgroups H_1 and H_2 such that the subgroups H_1 and H_2 are isomorphic while the factor groups G/H_1 and G/H_2 are not.

Problem 10. Complete the following Cayley table of a group of order 9:

$*$	A	B	C	D	E	F	G	H	I
A	I								F
B		F						G	
C			H				E		
D				G		A			
E					E				
F				A		B			
G			E				A		
H		G						D	
I	F								C