MATH 415 Spring 2021

Homework assignment #5

Problem 1. Prove that a subgroup H of a group G is normal (that is, $gH = Hg$ for all $g \in G$) if and only if $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.

Problem 2. Let $\phi : G \to H$ be a homomorphism of groups. Prove that ϕ is injective if and only if its kernel $\mathrm{Ker}(\phi)$ is trivial.

Problem 3. Suppose G is an abelian group and let $F(G)$ be the set of all elements of finite order in G. Prove that $F(G)$ is a subgroup of G.

Problem 4. Given a group G, an element $c \in G$ is called *central* if it commutes with any other element: $cg = gc$ for all $g \in G$. The set of all central elements is called the *center* of G and denoted $Z(G)$. Prove that $Z(G)$ is a normal subgroup of G.

Problem 5. Given two elements g and h of a group G, the element $[g, h] = ghg^{-1}h^{-1}$ is called their *commutator*. The subgroup of G generated by all commutators is called the *commutator* (or *derived*) *group* of G and denoted $[G, G]$ (or G'). Prove that $[G, G]$ is a normal subgroup of G.

Problem 6. Prove that the commutator group of the symmetric group S_n is the alternating group A_n. [Hint: show that the product of any two transpositions is a commutator.]

Problem 7 (2 pts). A group G is called *perfect* if $[G, G] = G$. Prove that the alternating group A_n is perfect for $n \geq 5$ (without using the fact that it is simple).

Problem 8 (2 pts). Suppose that a group G has two normal subgroups H_1 and H_2 such that $H_1 \cap H_2 = \{e\}$ and G is generated by these subgroups: $G = \langle H_1 \cup H_2 \rangle$. Prove that $G \cong H_1 \times H_2$. [Hint: consider a map $\phi : H_1 \times H_2 \to G$ given by $\phi(h_1, h_2) = h_1 h_2$.]