## Sample problems for Exam 2

Any problem may be altered, removed or replaced by a different one!

Problem 1. Let $M$ be the set of all $2 \times 2$ matrices of the form $\left(\begin{array}{cc}n & k \\ 0 & n\end{array}\right)$, where $n$ and $k$ are rational numbers. Under the operations of matrix addition and multiplication, does this set form a ring? Does $M$ form a field?

Problem 2. Let $L$ be the set of the following $2 \times 2$ matrices with entries from the field $\mathbb{Z}_{2}$ :

$$
A=\left(\begin{array}{cc}
{[0]} & {[0]} \\
{[0]} & {[0]}
\end{array}\right), \quad B=\left(\begin{array}{cc}
{[1]} & {[0]} \\
{[0]} & {[1]}
\end{array}\right), \quad C=\left(\begin{array}{cc}
{[1]} & {[1]} \\
{[1]} & {[0]}
\end{array}\right), \quad D=\left(\begin{array}{cc}
{[0]} & {[1]} \\
{[1]} & {[1]}
\end{array}\right) .
$$

Under the operations of matrix addition and multiplication, does this set form a ring? Does $L$ form a field?

Problem 3. Prove that for a ring with unity, commutativity of addition follows from the other axioms. [Hint: simplify the expression $(1+1)(x+y)$ in two different ways.]

Problem 4. Find a direct product of cyclic groups that is isomorphic to $G_{16}$ (multiplicative group of all invertible elements of the ring $\mathbb{Z}_{16}$ ).

Problem 5. Determine the last two digits of $303^{303}$.
Problem 6. Find all integer solutions of the equation $21 x-32 y=4$.
Problem 7. Find all integer solutions of the equation $2 x+3 y+5 z=7$.
Problem 8. Solve the equation $2 x^{100}+x^{71}+x^{29}=0$ over the field $\mathbb{Z}_{11}$.
Problem 9. Factor a polynomial $p(x)=x^{3}-3 x^{2}+3 x-2$ into irreducible factors over the field $\mathbb{Z}_{7}$.

Problem 10. Factor a polynomial $p(x)=x^{4}+x^{3}-2 x^{2}+3 x-1$ into irreducible factors over the field $\mathbb{Q}$. [Hint: since $p$ has integer coefficients, there exists a factorization such that each factor has integer coefficients.]

