Sample problems for the final exam

Any problem may be altered or replaced by a different one!

Problem 1. For any positive integer n let $n\mathbb{Z}$ denote the set of all integers divisible by n.
(i) Does the set $3\mathbb{Z}\cup 4\mathbb{Z}\cup 7\mathbb{Z}$ form a semigroup under addition? Does it form a group?
(ii) Does the set $3\mathbb{Z}\cup 4\mathbb{Z}\cup 7\mathbb{Z}$ form a semigroup under multiplication? Does it form a group?

Problem 2. Consider a relation $\sim$ on a group G defined as follows. For any $g, h \in G$ we let $g \sim h$ if and only if g is conjugate to h, which means that $g = xhx^{-1}$ for some $x \in G$ (where x may depend on g and h). Show that $\sim$ is an equivalence relation on G.

Problem 3. Find all subgroups of the group G_{15} (multiplicative group of invertible congruence classes modulo 15.)

Problem 4. Let $\pi = (1\,2)(2\,3)(3\,4)(4\,5)(5\,6)$, $\sigma = (1\,2\,3)(2\,3\,4)(3\,4\,5)(4\,5\,6)$. Find the order and the sign of the following permutations: π, σ, $\pi\sigma$, and $\sigma\pi$.

Problem 5. Let G be a group. Suppose H is a subgroup of G of finite index $(G : H)$. Further suppose that K is a subgroup of H of finite index $(H : K)$. Prove that K is a subgroup of finite index in G and, moreover, $(G : K) = (G : H)(H : K)$.

Problem 6. Let G be the group of all symmetries of a regular tetrahedron T. The group G naturally acts on the set of vertices of T, the set of edges of T, and the set of faces of T.
(i) Show that each of the three actions is transitive.
(ii) Show that the stabilizer of any vertex is isomorphic to the symmetric group S_3.
(iii) Show that the stabilizer of any edge is isomorphic to the Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$.
(iv) Show that the stabilizer of any face is isomorphic to S_3.

Problem 7. Let S be a nonempty set and $\mathcal{P}(S)$ be the set of all subsets of S.
(i) Prove that $\mathcal{P}(S)$ with the operations of symmetric difference $\triangle$ (as addition) and intersection $\cap$ (as multiplication) is a commutative ring with unity.
(ii) Prove that the ring $\mathcal{P}(S)$ is isomorphic to the ring of functions $\mathcal{F}(S, \mathbb{Z}_2)$.

Problem 8. Solve a system of congruences (find all solutions):

$$\begin{cases} x \equiv 2 \bmod 5, \\ x \equiv 3 \bmod 6, \\ x \equiv 6 \bmod 7. \end{cases}$$

Problem 9. Find all integer solutions of a system

$$\begin{cases} 2x + 5y - z = 1, \\ x - 2y + 3z = 2. \end{cases}$$

[Hint: eliminate one of the variables.]

Problem 10. Factor a polynomial $p(x) = x^4 - 2x^3 - x^2 - 2x + 1$ into irreducible factors over the fields $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_5$ and $\mathbb{Z}_7$.

[Hint: notice that $p(x) = x^4 p(1/x)$.]

Problem 11. Let

$$M = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\}, \quad J = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \,\middle|\, y \in \mathbb{R} \right\}.$$

(i) Show that M is a subring of the matrix ring $\mathcal{M}_{2,2}(\mathbb{R})$.
(ii) Show that J is a two-sided ideal in M.
(iii) Show that the factor ring M/J is isomorphic to $\mathbb{R} \times \mathbb{R}$.

Problem 12. The polynomial $f(x) = x^6 + 3x^5 - 5x^3 + 3x - 1$ has how many distinct complex roots?

[Hint: multiple roots of f are also roots of the derivative f'.]