

## Homework assignment #5

**Problem 1.** Prove that a subgroup  $H$  of a group  $G$  is normal (that is,  $gH = Hg$  for all  $g \in G$ ) if and only if  $ghg^{-1} \in H$  for all  $g \in G$  and  $h \in H$ .

**Problem 2.** Let  $\phi : G \rightarrow H$  be a homomorphism of groups. Prove that  $\phi$  is injective if and only if its kernel  $\text{Ker}(\phi)$  is trivial.

**Problem 3.** Show that the relation “is a normal subgroup of” is not transitive. In other words, find a group  $G$  with two subgroups  $H$  and  $K$  such that  $K$  is a normal subgroup of  $H$ ,  $H$  is a normal subgroup of  $G$ , but  $K$  is not a normal subgroup of  $G$ .

**Problem 4 (2 pts).** Given an abelian group  $G$ , let  $F(G)$  be the set of all elements of finite order in  $G$ .

- (i) Prove that  $F(G)$  is a subgroup of  $G$ .
- (ii) Describe  $F(G)$  in the case when  $G = \mathbb{R}/\mathbb{Z}$ .

**Problem 5.** Given a group  $G$ , an element  $c \in G$  is called *central* if it commutes with any other element:  $cg = gc$  for all  $g \in G$ . The set of all central elements is called the *center* of  $G$  and denoted  $Z(G)$ . Prove that  $Z(G)$  is a normal subgroup of  $G$ .

**Problem 6.** Given two elements  $g$  and  $h$  of a group  $G$ , the element  $[g, h] = ghg^{-1}h^{-1}$  is called their *commutator*. The subgroup of  $G$  generated by all commutators is called the *commutator* (or *derived*) *group* of  $G$  and denoted  $[G, G]$  (or  $G'$ ). Prove that  $[G, G]$  is a normal subgroup of  $G$ .

**Problem 7.** Prove that the commutator group of the symmetric group  $S_n$  is the alternating group  $A_n$ . [Hint: show that the product of any two transpositions is a commutator.]

**Problem 8 (2 pts).** A group  $G$  is called *perfect* if  $[G, G] = G$ . Prove that the alternating group  $A_n$  is perfect for  $n \geq 5$  (without using the fact that it is simple).