MATH 415
Modern Algebra I

Lecture 2:
Cardinality of a set.

Functions

A **function** (or **map**) $f : X \to Y$ is an assignment: to each $x \in X$ we assign an element $f(x) \in Y$.

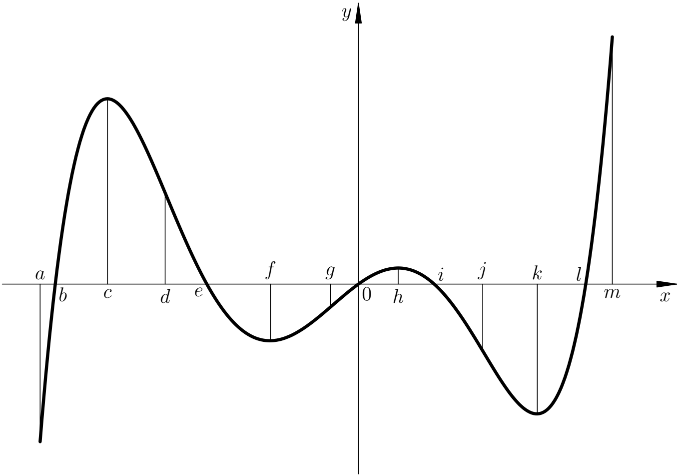
Definition. A function $f : X \to Y$ is **injective** (or **one-to-one**) if $f(x') = f(x) \implies x' = x$.

The function f is **surjective** (or **onto**) if for each $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$.

Finally, f is **bijective** if it is both surjective and injective. Equivalently, if for each $y \in Y$ there is exactly one $x \in X$ such that $f(x) = y$.

Suppose we have two functions $f : X \to Y$ and $g : Y \to X$. We say that g is the **inverse function** of f (denoted f^{-1}) if $y = f(x) \iff g(y) = x$ for all $x \in X$ and $y \in Y$.

Theorem The inverse function f^{-1} exists if and only if f is bijective.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Definition. The **composition** of functions $f : X \to Y$ and $g : Y \to Z$ is a function from X to Z, denoted $g \circ f$, that is defined by $(g \circ f)(x) = g(f(x))$, $x \in X$.

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

Properties of compositions:

- If f and g are one-to-one, then $g \circ f$ is also one-to-one.
- If $g \circ f$ is one-to-one, then f is also one-to-one.
- If f and g are onto, then $g \circ f$ is also onto.
- If $g \circ f$ is onto, then g is also onto.
- If f and g are bijective, then $g \circ f$ is also bijective.
- If f and g are invertible, then $g \circ f$ is also invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
- If id_Z denotes the identity function on a set Z, then $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$ for any function $f : X \to Y$.
- For any functions $f : X \to Y$ and $g : Y \to X$, we have $g = f^{-1}$ if and only if $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Cardinality of a set

Definition. Given two sets A and B, we say that A is of the same **cardinality** as B if there exists a bijective function $f : A \to B$. Notation: $|A| = |B|$.

Theorem The relation "is of the same cardinality as" is an equivalence relation, i.e., it is reflexive ($|A| = |A|$ for any set A), symmetric ($|A| = |B|$ implies $|B| = |A|$), and transitive ($|A| = |B|$ and $|B| = |C|$ imply $|A| = |C|$).

Proof: The identity map $\mathrm{id}_A : A \to A$ is bijective. If f is a bijection of A onto B, then the inverse map f^{-1} is a bijection of B onto A. If $f : A \to B$ and $g : B \to C$ are bijections then the composition $g \circ f$ is a bijection of A onto C.

Countable and uncountable sets

A nonempty set is **finite** if it is of the same cardinality as $\{1, 2, \ldots, n\} = [1, n] \cap \mathbb{N}$ for some $n \in \mathbb{N}$. Otherwise it is **infinite**.

An infinite set is called **countable** (or **countably infinite**) if it is of the same cardinality as $\mathbb{N}$. Otherwise it is **uncountable** (or **uncountably infinite**).

An infinite set E is countable if it is possible to arrange all elements of E into a single sequence (an infinite list) $x_1, x_2, x_3, \ldots$ The sequence is referred to as an **enumeration** of E.

Countable sets

- $2\mathbb{N}$: even natural numbers.

Bijection $f : \mathbb{N} \to 2\mathbb{N}$ is given by $f(n) = 2n$.

- $\mathbb{N} \cup \{0\}$: nonnegative integers.

Bijection $f : \mathbb{N} \to \mathbb{N} \cup \{0\}$ is given by $f(n) = n - 1$.

- $\mathbb{Z}$: integers.

Enumeration of all integers: $0, 1, -1, 2, -2, 3, -3, \dots$
Equivalently, a bijection $f : \mathbb{N} \to \mathbb{Z}$ is given by $f(n) = n/2$ if n is even and $f(n) = (1 - n)/2$ if n is odd.

- $E_1 \cup E_2$, where E_1 is finite and E_2 is countable.

First we list all elements of E_1. Then we append the list of all elements of E_2. If E_1 and E_2 are not disjoint, we also need to avoid repetitions in the joint list.

Countable sets

- $E_1 \cup E_2$, where E_1 and E_2 are countable.

Let $x_1, x_2, x_3 \ldots$ be an enumeration of E_1 and $y_1, y_2, y_3, \ldots$ be an enumeration of E_2. Then $x_1, y_1, x_2, y_2, \ldots$ enumerates the union (maybe with repetitions).

- Infinite set $E_1 \cup E_2 \cup \ldots,$ where each E_n is finite.

First we list all elements of E_1. Then we append the list of all elements of E_2. Then we append the list of all elements of E_3, and so on... (and do not forget to avoid repetitions).

- $\mathbb{N} \times \mathbb{N}$: pairs of natural numbers

- $\mathbb{Q}$: rational numbers

- Algebraic numbers (roots of nonzero polynomials with integer coefficients).

Theorem (Cantor) The set $\mathbb{R}$ is uncountable.

Proof: It is enough to prove that the interval $(0,1)$ is uncountable. Assume the contrary. Then all numbers from $(0,1)$ can be arranged into an infinite list $x_1, x_2, x_3, \ldots$ Any number $x \in (0,1)$ admits a decimal expansion of the form $0.d_1 d_2 d_3 \ldots$, where each $d_i \in \{0, 1, \ldots, 9\}$. In particular,

$$x_1 = 0.d_{11} d_{12} d_{13} d_{14} d_{15} \ldots$$
$$x_2 = 0.d_{21} d_{22} d_{23} d_{24} d_{25} \ldots$$
$$x_3 = 0.d_{31} d_{32} d_{33} d_{34} d_{35} \ldots$$

. . . .

Now for any $n \in \mathbb{N}$ choose a decimal digit $\tilde{d}_n$ such that $\tilde{d}_n \neq d_{nn}$ and $\tilde{d}_n \notin \{0, 9\}$. Then $0.\tilde{d}_1 \tilde{d}_2 \tilde{d}_3 \ldots$ is the decimal expansion of some number $\tilde{x} \in (0,1)$. By construction, it is different from all expansions in the list. Although some real numbers admit two decimal expansions (e.g., $0.50000\ldots$ and $0.49999\ldots$), the condition $\tilde{d}_n \notin \{0, 9\}$ ensures that $\tilde{x}$ is not such a number. Thus $\tilde{x}$ is not listed, a contradiction.

Uncountable sets

- Any interval (a, b) is of the same cardinality as $(0, 1)$.

Bijection $f : (0, 1) \to (a, b)$ is given by $f(x) = (b - a)x + a$.

- All intervals of the form (a, b) have the same cardinality.

Follows by transitivity since they are all of the same cardinality as $(0, 1)$.

- All intervals of the form (a, ∞) or $(-\infty, a)$ are of the same cardinality as $(0, \infty)$.

Bijection $f : (0, \infty) \to (a, \infty)$ is given by $f(x) = x + a$.
Bijection $f : (0, \infty) \to (-\infty, a)$ is given by $f(x) = -x + a$.

Uncountable sets

- $(0, 1)$ is of the same cardinality as $(1, \infty)$.

Bijection $f : (0, 1) \to (1, \infty)$ is given by $f(x) = x^{-1}$.

- $(0, \infty)$ is of the same cardinality as $\mathbb{R}$.

Bijection $f : \mathbb{R} \to (0, \infty)$ is given by $f(x) = e^x$.

- $[0, 1]$ is of the same cardinality as $(0, 1)$.

Let $x_1, x_2, x_3, \dots$ be a sequence of distinct points in $(0, 1)$, say, $x_n = (n + 1)^{-1}$ for all $n \in \mathbb{N}$. Then a bijection $f : [0, 1] \to (0, 1)$ is defined as follows: $f(0) = x_1$, $f(1) = x_2$, $f(x_n) = x_{n+2}$ for all $n \in \mathbb{N}$, and $f(x) = x$ otherwise.

How to compare cardinalities?

Definition. Given two sets A and B, we say that the cardinality of A is **less than or equal to** the cardinality of B (and write $|A| \preceq |B|$) if the set A is of the same cardinality as some subset of B. An equivalent condition is that there exists an injective function $f : A \to B$.

We say that the cardinality of A is **less than** the cardinality of B (and write $|A| \prec |B|$) if $|A| \preceq |B|$ and $|A| \neq |B|$.

Proposition (i) If $|A| \preceq |B|$ and $|B| \preceq |C|$, then $|A| \preceq |C|$.
(ii) If $|A| \prec |B|$ and $|B| \prec |C|$, then $|A| \prec |C|$.

Theorem (Schröder-Bernstein) If $|A| \preceq |B|$ and $|B| \preceq |A|$, then $|A| = |B|$.

Hence $\preceq$ (or $\prec$) is an ordering of cardinalities. Moreover, this ordering is **total**, i.e., any two cardinalities are comparable.

Theorem For any two sets A and B, we have either $|A| \prec |B|$ or $|B| \prec |A|$ or $|A| = |B|$.