MATH 415

Modern Algebra I

Lecture 11:
Sign of a permutation.
Classical definition of the determinant.

Sign of a permutation

Theorem 1 (i) Any permutation of $n \geq 2$ elements is a product of transpositions. **(ii)** If $\pi = \tau_1 \tau_2 \ldots \tau_k = \tau'_1 \tau'_2 \ldots \tau'_m$, where τ_i, τ'_j are transpositions, then the numbers k and m are of the same parity (that is, both even or both odd).

A permutation π is called **even** if it is a product of an even number of transpositions, and **odd** if it is a product of an odd number of transpositions.

The **sign** $\mathrm{sgn}(\pi)$ of the permutation π is defined to be $+1$ if π is even, and -1 if π is odd.

Theorem 2 (i) $\mathrm{sgn}(\pi\sigma) = \mathrm{sgn}(\pi)\,\mathrm{sgn}(\sigma)$ for any $\pi, \sigma \in S_n$.
(ii) $\mathrm{sgn}(\pi^{-1}) = \mathrm{sgn}(\pi)$ for any $\pi \in S_n$.
(iii) $\mathrm{sgn}(\mathrm{id}) = 1$.
(iv) $\mathrm{sgn}(\tau) = -1$ for any transposition τ.
(v) $\mathrm{sgn}(\sigma) = (-1)^{r-1}$ for any cycle σ of length r.

Let $\pi \in S_n$ and i, j be integers, $1 \leq i < j \leq n$. We say that the permutation π preserves order of the pair (i, j) if $\pi(i) < \pi(j)$. Otherwise π makes an **inversion**. Denote by $N(\pi)$ the number of inversions made by the permutation π.

Lemma 1 Let $\tau, \pi \in S_n$ and suppose that τ is an adjacent transposition, $\tau = (k\ k{+}1)$. Then $|N(\tau\pi) - N(\pi)| = 1$.

Proof: For every pair (i, j), $1 \leq i < j \leq n$, let us compare the order of pairs $\pi(i), \pi(j)$ and $\tau\pi(i), \tau\pi(j)$. We observe that the order differs exactly for one pair, when $\{\pi(i), \pi(j)\} = \{k, k{+}1\}$. The lemma follows.

Lemma 2 Let $\pi \in S_n$ and $\tau_1, \tau_2, \ldots, \tau_k$ be adjacent transpositions. Then **(i)** for any $\pi \in S_n$ the numbers k and $N(\tau_1\tau_2\ldots\tau_k\pi) - N(\pi)$ are of the same parity,

(ii) the numbers k and $N(\tau_1\tau_2\ldots\tau_k)$ are of the same parity.

Sketch of the proof: **(i)** follows from Lemma 1 by induction on k. **(ii)** is a particular case of part (i), when $\pi = \mathrm{id}$.

Lemma 3 (i) Any cycle of length r is a product of $r-1$ transpositions. **(ii)** Any transposition is a product of an odd number of adjacent transpositions.

Proof: **(i)** $(x_1\ x_2\ \dots\ x_r) = (x_1\ x_2)(x_2\ x_3)(x_3\ x_4)\dots(x_{r-1}\ x_r)$.
(ii) $(k\ k{+}r) = \sigma^{-1}(k\ k{+}1)\sigma$, where $\sigma = (k{+}1\ k{+}2\ \dots\ k{+}r)$.
By the above, $\sigma = (k{+}1\ k{+}2)(k{+}2\ k{+}3)\dots(k{+}r{-}1\ \ k{+}r)$ and $\sigma^{-1} = (k{+}r\ \ k{+}r{-}1)\dots(k{+}3\ k{+}2)(k{+}2\ k{+}1)$.

Theorem (i) Any permutation is a product of transpositions. **(ii)** If $\pi = \tau_1\tau_2\dots\tau_k$, where τ_i are transpositions, then the numbers k and $N(\pi)$ are of the same parity.

Proof: **(i)** Any permutation is a product of disjoint cycles. By Lemma 3, any cycle is a product of transpositions.

(ii) By Lemma 3, each of $\tau_1, \tau_2, \dots, \tau_k$ is a product of an odd number of adjacent transpositions. Hence $\pi = \tau'_1\tau'_2\dots\tau'_m$, where τ'_i are adjacent transpositions and number m is of the same parity as k. By Lemma 2, m has the same parity as $N(\pi)$.

Examples

- $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 4 & 7 & 9 & 1 & 12 & 5 & 11 & 3 & 10 & 6 & 8 \end{pmatrix}.$

First we decompose π into a product of disjoint cycles:

$$\pi = (1\ 2\ 4\ 9\ 3\ 7\ 5)(6\ 12\ 8\ 11).$$

The cycle $\sigma_1 = (1\ 2\ 4\ 9\ 3\ 7\ 5)$ has length 7, hence it is an even permutation. The cycle $\sigma_2 = (6\ 12\ 8\ 11)$ has length 4, hence it is an odd permutation. Then

$$\operatorname{sgn}(\pi) = \operatorname{sgn}(\sigma_1\sigma_2) = \operatorname{sgn}(\sigma_1)\operatorname{sgn}(\sigma_2) = 1 \cdot (-1) = -1.$$

- $\pi = (2\ 4\ 3)(1\ 2)(2\ 3\ 4).$

π is represented as a product of cycles. The transposition has sign -1 while the cycles of length 3 have sign $+1$. Even though the cycles are not disjoint, $\operatorname{sgn}(\pi) = 1 \cdot (-1) \cdot 1 = -1$.

Theorem The symmetric group S_n is generated by two permutations: $\tau = (1\ 2)$ and $\pi = (1\ 2\ 3\ \dots\ n)$.

Proof: Let $H = \langle \tau, \pi \rangle$. We have to show that $H = S_n$.

First we obtain that $\alpha = \tau\pi = (2\ 3\ \dots\ n)$. Then we observe that $\sigma(1\ 2)\sigma^{-1} = (\sigma(1)\ \sigma(2))$ for any permutation σ.
In particular, $(1\ k) = \alpha^{k-2}(1\ 2)(\alpha^{k-2})^{-1}$ for $k = 2, 3 \dots, n$.
It follows that the subgroup H contains all transpositions of the form $(1\ k)$. Further, for any integers $2 \le k < m \le n$ we have $(k\ m) = (1\ k)(1\ m)(1\ k)$. Therefore the subgroup H contains all transpositions.

Next, every cycle of length $r \ge 2$ is a product of $r - 1$ transpositions. Indeed,

$$(x_1\ x_2\ \dots\ x_r) = (x_1\ x_2)(x_2\ x_3)(x_3\ x_4)\dots(x_{r-1}\ x_r).$$

Hence the subgroup H contains all cycles.

Finally, every permutation in S_n is a product of cycles, therefore it is contained in H. Thus $H = S_n$.

Alternating groups

Given an integer $n \geq 2$, the **alternating group** on n symbols, denoted A_n or $A(n)$, is the set of all even permutations in the symmetric group S_n.

Theorem The alternating group A_n is a subgroup of the symmetric group S_n.

In other words, the product of even permutations is even, the identity function is an even permutation, and the inverse of an even permutation is even.

Theorem The alternating group A_n has $n!/2$ elements.

Proof: Consider the function $F : A_n \to S_n \setminus A_n$ given by $F(\pi) = (1\ 2)\pi$. One can observe that F is bijective. Hence the sets A_n and $S_n \setminus A_n$ have the same number of elements.

Examples. • The alternating group A_3 has 3 elements: the identity function and two cycles of length 3, $(1\ 2\ 3)$ and $(1\ 3\ 2)$.

• The alternating group A_4 has 12 elements of the following **cycle shapes**: id, $(1\ 2\ 3)$, and $(1\ 2)(3\ 4)$.

• The alternating group A_5 has 60 elements of the following cycle shapes: id, $(1\ 2\ 3)$, $(1\ 2)(3\ 4)$, and $(1\ 2\ 3\ 4\ 5)$.

Classical definition of the determinant

Definition. $\det(a) = a,\quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

If $A = (a_{ij})$ is an $n \times n$ matrix then

$$\det A = \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\, a_{1,\pi(1)}\, a_{2,\pi(2)} \dots a_{n,\pi(n)},$$

where π runs over all permutations of $\{1, 2, \dots, n\}$.

Theorem $\det A^T = \det A$.

Proof: Let $A = (a_{ij})_{1\le i,j\le n}$. Then $A^T = (b_{ij})_{1\le i,j\le n}$, where $b_{ij} = a_{ji}$. We have

$$\det A^T = \sum_{\pi\in S_n} \operatorname{sgn}(\pi)\, b_{1,\pi(1)}\, b_{2,\pi(2)} \dots b_{n,\pi(n)}$$
$$= \sum_{\pi\in S_n} \operatorname{sgn}(\pi)\, a_{\pi(1),1}\, a_{\pi(2),2} \dots a_{\pi(n),n}$$
$$= \sum_{\pi\in S_n} \operatorname{sgn}(\pi)\, a_{1,\pi^{-1}(1)}\, a_{2,\pi^{-1}(2)} \dots a_{n,\pi^{-1}(n)}.$$

When π runs over all permutations of $\{1, 2, \dots, n\}$, so does $\sigma = \pi^{-1}$. It follows that

$$\det A^T = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma^{-1})\, a_{1,\sigma(1)}\, a_{2,\sigma(2)} \dots a_{n,\sigma(n)}$$
$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, a_{1,\sigma(1)}\, a_{2,\sigma(2)} \dots a_{n,\sigma(n)} = \det A.$$

Theorem 1 Suppose A is a square matrix and B is obtained from A by exchanging two rows. Then $\det B = -\det A$.

Theorem 2 Suppose A is a square matrix and B is obtained from A by permuting its rows. Then $\det B = \det A$ if the permutation is even and $\det B = -\det A$ if the permutation is odd.

Proof: Let $A = (a_{ij})_{1 \le i,j \le n}$ be an $n \times n$ matrix. Suppose that a matrix B is obtained from A by permuting its rows according to a permutation $\sigma \in S_n$. Then $B = (b_{ij})_{1 \le i,j \le n}$, where $b_{\sigma(i),j} = a_{ij}$. Equivalently, $b_{ij} = a_{\sigma^{-1}(i),j}$. We have

$$\det B = \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\, b_{1,\pi(1)}\, b_{2,\pi(2)} \dots b_{n,\pi(n)}$$

$$= \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\, a_{\sigma^{-1}(1),\pi(1)}\, a_{\sigma^{-1}(2),\pi(2)} \dots a_{\sigma^{-1}(n),\pi(n)}$$

$$= \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\, a_{1,\pi\sigma(1)}\, a_{2,\pi\sigma(2)} \dots a_{n,\pi\sigma(n)}.$$

When π runs over all permutations of $\{1, 2, \dots, n\}$, so does $\tau = \pi\sigma$. It follows that

$$\det B = \sum_{\tau \in S_n} \operatorname{sgn}(\tau\sigma^{-1})\, a_{1,\tau(1)}\, a_{2,\tau(2)} \dots a_{n,\tau(n)}$$

$$= \operatorname{sgn}(\sigma^{-1}) \sum_{\tau \in S_n} \operatorname{sgn}(\tau)\, a_{1,\tau(1)}\, a_{2,\tau(2)} \dots a_{n,\tau(n)} = \operatorname{sgn}(\sigma) \det A.$$

The Vandermonde determinant

Definition. The **Vandermonde determinant** is the determinant of the following matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix},$$

where $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Equivalently, $V = (a_{ij})_{1 \le i,j \le n}$, where $a_{ij} = x_i^{j-1}$.

Theorem

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

Corollary Consider a polynomial

$$p(x_1, x_2, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

Then

$$p(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)}) = \mathrm{sgn}(\pi)\, p(x_1, x_2, \dots, x_n)$$

for any permutation $\pi \in S_n$.