MATH 415

Modern Algebra I

Lecture 39:
Review for the final exam.

Topics for the final exam

Group theory:

- Binary operations
- Groups, semigroups
- Subgroups, cyclic groups
- Groups of permutations
- Cosets, Lagrange's theorem

- Direct product of groups
- Factor groups
- Homomorphisms of groups
- Classification of abelian groups
- Group actions

Fraleigh/Brand: Sections 0–14

Topics for the final exam

Theory of rings and fields:

- Rings and fields
- Integral domains
- Modular arithmetic
- Rings of polynomials
- Factorization of polynomials

- Ideals
- Factor rings
- Homomorphisms of rings
- Prime and maximal ideals
- Euclidean algorithm

Fraleigh/Brand: Sections 22–24, 26–28, 30–32.

Sample problems

Problem 1. For any positive integer n let $n\mathbb{Z}$ denote the set of all integers divisible by n.

(i) Does the set $3\mathbb{Z} \cup 4\mathbb{Z} \cup 7\mathbb{Z}$ form a semigroup under addition? Does it form a group?

(ii) Does the set $3\mathbb{Z} \cup 4\mathbb{Z} \cup 7\mathbb{Z}$ form a semigroup under multiplication? Does it form a group?

Problem 2. Consider a relation $\sim$ on a group G defined as follows. For any $g, h \in G$ we let $g \sim h$ if and only if g is conjugate to h, which means that $g = xhx^{-1}$ for some $x \in G$ (where x may depend on g and h). Show that $\sim$ is an equivalence relation on G.

Problem 3. Find all subgroups of the group G_{15} (multiplicative group of invertible congruence classes modulo 15.)

Sample problems

Problem 4. Let $\pi = (1\,2)(2\,3)(3\,4)(4\,5)(5\,6)$ and $\sigma = (1\,2\,3)(2\,3\,4)(3\,4\,5)(4\,5\,6)$. Find the order and the sign of the following permutations: π, σ, $\pi\sigma$, and $\sigma\pi$.

Problem 5. Let G be a group. Suppose H is a subgroup of G of finite index $(G : H)$ and K is a subgroup of H of finite index $(H : K)$. Prove that K is a subgroup of finite index in G and, moreover, $(G : K) = (G : H)(H : K)$.

Problem 6. Let G be the group of all symmetries of a regular tetrahedron T. The group G naturally acts on the set of vertices of T, the set of edges of T, and the set of faces of T.

(i) Show that each of the three actions is transitive.

(ii) Show that the stabilizer of any vertex is isomorphic to the symmetric group S_3.

(iii) Show that the stabilizer of any edge is isomorphic to the Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(iv) Show that the stabilizer of any face is isomorphic to S_3.

Sample problems

Problem 7. Let S be a nonempty set and $\mathcal{P}(S)$ be the set of all subsets of S. **(i)** Prove that $\mathcal{P}(S)$ with the operations of symmetric difference $\triangle$ (as addition) and intersection $\cap$ (as multiplication) is a commutative ring with unity.

(ii) Prove that the ring $\mathcal{P}(S)$ is isomorphic to the ring of functions $\mathcal{F}(S, \mathbb{Z}_2)$.

Problem 8. Solve a system of congruences (find all solutions):

$$\begin{cases} x \equiv 2 \mod 5, \\ x \equiv 3 \mod 6, \\ x \equiv 6 \mod 7. \end{cases}$$

Problem 9. Find all integer solutions of a system

$$\begin{cases} 2x + 5y - z = 1, \\ x - 2y + 3z = 2. \end{cases}$$

Sample problems

Problem 10. Factor a polynomial $p(x) = x^4 - 2x^3 - x^2 - 2x + 1$ into irreducible factors over the fields $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_5$ and $\mathbb{Z}_7$.

Problem 11. Let

$$M = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}, \quad J = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \;\middle|\; y \in \mathbb{R} \right\}.$$

(i) Show that M is a subring of the matrix ring $\mathcal{M}_{2,2}(\mathbb{R})$.
(ii) Show that J is a two-sided ideal in M.
(iii) Show that the factor ring M/J is isomorphic to $\mathbb{R} \times \mathbb{R}$.

Problem 12. The polynomial $f(x) = x^6 + 3x^5 - 5x^3 + 3x - 1$ has how many distinct complex roots?

Problem 1. For any positive integer n let $n\mathbb{Z}$ denote the set of all integers divisible by n.

(i) Does the set $3\mathbb{Z} \cup 4\mathbb{Z} \cup 7\mathbb{Z}$ form a semigroup under addition? Does it form a group?

(ii) Does the set $3\mathbb{Z} \cup 4\mathbb{Z} \cup 7\mathbb{Z}$ form a semigroup under multiplication? Does it form a group?

The set $S = 3\mathbb{Z} \cup 4\mathbb{Z} \cup 7\mathbb{Z}$ consists of all integers divisible by at least one of the numbers 3, 4 and 7. This set is not closed under addition. For example, the numbers 4 and 7 belong to S while their sum $4 + 7 = 11$ does not. Therefore S is neither a semigroup nor a group with respect to addition.

Each of the sets $3\mathbb{Z}$, $4\mathbb{Z}$ and $7\mathbb{Z}$ is closed under multiplication by any integer. Hence their union S is also closed under multiplication by any integer. In particular, S is a semigroup with respect to multiplication. It is not a group since it does not contain 1 (and 1 is the only number that can be the multiplicative identity element unless $S = \{0\}$).

Problem 2. Consider a relation $\sim$ on a group G defined as follows. For any $g, h \in G$ we let $g \sim h$ if and only if g is conjugate to h, which means that $g = xhx^{-1}$ for some $x \in G$ (where x may depend on g and h). Show that $\sim$ is an equivalence relation on G.

We have to show that the relation $\sim$ is reflexive, symmetric and transitive.

Reflexivity. $g \sim g$ since $g = ege^{-1}$, where e is the identity element.

Symmetry. Assume $g \sim h$, that is, $g = xhx^{-1}$ for some $x \in G$. Then $h = x^{-1}gx = x^{-1}g(x^{-1})^{-1} = x_1 g x_1^{-1}$, where $x_1 = x^{-1}$. Hence $h \sim g$.

Transitivity. Assume $g \sim h$ and $h \sim k$, that is, $g = x_1 h x_1^{-1}$ and $h = x_2 k x_2^{-1}$ for some $x_1, x_2 \in G$. Then $g = x_1(x_2 k x_2^{-1})x_1^{-1} = (x_1 x_2)k(x_2^{-1}x_1^{-1}) = (x_1 x_2)k(x_1 x_2)^{-1}$ $= xkx^{-1}$, where $x = x_1 x_2$. Hence $g \sim k$.

Problem 3. Find all subgroups of the group G_{15} (multiplicative group of invertible congruence classes modulo 15.)

A congruence class $[a]_{15}$ belongs to G_{15} if and only if $\gcd(a, 15) = 1$. Hence the group G_{15} consists of the following 8 elements: [1], [2], [4], [7], [8], [11], [13], [14] or, equivalently, [1], [2], [4], [7], [−7], [−4], [−2], [−1].

First we find all cyclic subgroups of G_{15}. These are $\{[1]\}$, $\{[1],[4]\}$, $\{[1],[-4]\}$, $\{[1],[-1]\}$, $\{[1],[2],[4],[8]\}$, and $\{[1],[4],[7],[13]\} = \{[1],[-2],[4],[-8]\}$.

Note that any subgroup of G_{15} is a union of (one or more) cyclic subgroups. By Lagrange's Theorem, a subgroup of G_{15} can be of order 1, 2, 4 or 8. It follows that the only possible non-cyclic subgroups of G_{15} might be G_{15} itself and $\{[1],[4],[-4],[-1]\}$. We can check that both are indeed subgroups.

Remark. $G_{15} \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.

Problem 4. Let $\pi = (1\,2)(2\,3)(3\,4)(4\,5)(5\,6)$ and $\sigma = (1\,2\,3)(2\,3\,4)(3\,4\,5)(4\,5\,6)$. Find the order and the sign of the following permutations: π, σ, $\pi\sigma$, and $\sigma\pi$.

Any transposition is an odd permutation, its sign is -1. Any cycle of length 3 is an even permutation, its sign is $+1$. Since the sign is a multiplicative function, we obtain that $\mathrm{sgn}(\pi) = (-1)^5 = -1$ and $\mathrm{sgn}(\sigma) = 1^4 = 1$. Then $\mathrm{sgn}(\pi\sigma) = \mathrm{sgn}(\sigma\pi) = \mathrm{sgn}(\pi)\,\mathrm{sgn}(\sigma) = -1$.

To find the order of a permutation, we need to decompose it into a product of disjoint cycles. First we decompose π and σ: $\pi = (1\,2\,3\,4\,5\,6)$, $\sigma = (1\,2)(5\,6)$. Then we use these decompositions to decompose $\pi\sigma$ and $\sigma\pi$: $\pi\sigma = (1\,3\,4\,5)$ and $\sigma\pi = (2\,3\,4\,6)$. The order of a product of disjoint cycles equals the least common multiple of their lengths. Therefore $o(\pi) = 6$, $o(\sigma) = 2$, and $o(\pi\sigma) = o(\sigma\pi) = 4$.

Problem 5. Let G be a group. Suppose H is a subgroup of G of finite index $(G : H)$ and K is a subgroup of H of finite index $(H : K)$. Prove that K is a subgroup of finite index in G and, moreover, $(G : K) = (G : H)(H : K)$.

First assume G is a finite group. Then any subgroup is of finite order and of finite index. By Lagrange's Theorem, $|G| = (G : H)\,|H|$ and $|H| = (H : K)\,|K|$ so that $|G| = (G : H)(H : K)\,|K|$. Also by Lagrange's Theorem, $|G| = (G : K)\,|K|$. It follows that $(G : K) = (G : H)(H : K)$.

In the general case, we need a different argument.

Problem 5. Let G be a group. Suppose H is a subgroup of G of finite index $(G : H)$ and K is a subgroup of H of finite index $(H : K)$. Prove that K is a subgroup of finite index in G and, moreover, $(G : K) = (G : H)(H : K)$.

Let $n = (G : H)$ and suppose $g_1, g_2, \ldots, g_n$ is a complete list of representatives of the left cosets of H in G. Further, let $k = (H : K)$ and suppose $h_1, h_2, \ldots, h_k$ is a complete list of representatives of the left cosets of K in H. Then G is a disjoint union of cosets $g_1H, g_2H, \ldots, g_nH$ while H is a disjoint union of cosets $h_1K, h_2K, \ldots, h_kK$. It follows that each g_iH is a disjoint union of sets $g_ih_1K, g_ih_2K, \ldots, g_ih_kK$, which are cosets of K in G. Therefore G is a disjoint union of all sets of the form g_ih_jK, $1 \leq i \leq n$, $1 \leq j \leq k$. Hence these are all cosets of the subgroup K in G. Thus the number $(G : K)$ of the cosets equals $nk = (G : H)(H : K)$.

Problem 6. Let G be the group of all symmetries of a regular tetrahedron T. The group G naturally acts on the set of vertices of T, the set of edges of T, and the set of faces of T.

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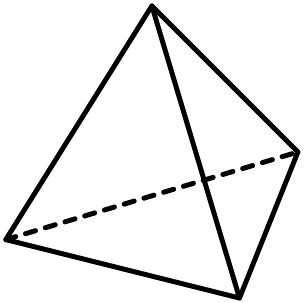

(i) Show that each of the three actions is transitive.

We can label vertices of T by 1, 2, 3 and 4. Then the action of G on the vertices induces a homomorphism $h : G \to S_4$ (permutation representation). This homomorphism is injective since any isometry of $\mathbb{R}^3$ is uniquely determined by images of any 4 points not in the same plane. Observe that every transposition is in the image $h(G)$ (it is realized by a reflection about a plane of symmetry of T). Since the symmetric group S_4 is generated by transpositions, it follows that $h(G) = S_4$. Hence h is an isomorphism.

In view of the isomorphism h, the action of G on vertices of T is essentially the natural action of S_4 on $\{1, 2, 3, 4\}$. Since any two vertices of T are endpoints of a unique edge and any three vertices are vertices of a unique face, the actions of G on edges and vertices of T are essentially the actions of S_4 on two-element and three-element subsets of $\{1, 2, 3, 4\}$. Transitivity of all three actions follows.