MATH 415

Modern Algebra I

Lecture 40: Review for the final exam (continued).

Sample problems

Problem 1. For any positive integer n let $n\mathbb{Z}$ denote the set of all integers divisible by n.

(i) Does the set $3\mathbb{Z} \cup 4\mathbb{Z} \cup 7\mathbb{Z}$ form a semigroup under addition? Does it form a group?

(ii) Does the set $3\mathbb{Z} \cup 4\mathbb{Z} \cup 7\mathbb{Z}$ form a semigroup under multiplication? Does it form a group?

Problem 2. Consider a relation $\sim$ on a group G defined as follows. For any $g, h \in G$ we let $g \sim h$ if and only if g is conjugate to h, which means that $g = xhx^{-1}$ for some $x \in G$ (where x may depend on g and h). Show that $\sim$ is an equivalence relation on G.

Problem 3. Find all subgroups of the group G_{15} (multiplicative group of invertible congruence classes modulo 15.)

Sample problems

Problem 4. Let $\pi = (1\,2)(2\,3)(3\,4)(4\,5)(5\,6)$ and $\sigma = (1\,2\,3)(2\,3\,4)(3\,4\,5)(4\,5\,6)$. Find the order and the sign of the following permutations: π, σ, $\pi\sigma$, and $\sigma\pi$.

Problem 5. Let G be a group. Suppose H is a subgroup of G of finite index $(G : H)$ and K is a subgroup of H of finite index $(H : K)$. Prove that K is a subgroup of finite index in G and, moreover, $(G : K) = (G : H)(H : K)$.

Problem 6. Let G be the group of all symmetries of a regular tetrahedron T. The group G naturally acts on the set of vertices of T, the set of edges of T, and the set of faces of T.

(i) Show that each of the three actions is transitive.

(ii) Show that the stabilizer of any vertex is isomorphic to the symmetric group S_3.

(iii) Show that the stabilizer of any edge is isomorphic to the Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(iv) Show that the stabilizer of any face is isomorphic to S_3.

Sample problems

Problem 7. Let S be a nonempty set and $\mathcal{P}(S)$ be the set of all subsets of S. **(i)** Prove that $\mathcal{P}(S)$ with the operations of symmetric difference $\triangle$ (as addition) and intersection $\cap$ (as multiplication) is a commutative ring with unity.

(ii) Prove that the ring $\mathcal{P}(S)$ is isomorphic to the ring of functions $\mathcal{F}(S, \mathbb{Z}_2)$.

Problem 8. Solve a system of congruences (find all solutions):

$$\begin{cases} x \equiv 2 \bmod 5, \\ x \equiv 3 \bmod 6, \\ x \equiv 6 \bmod 7. \end{cases}$$

Problem 9. Find all integer solutions of a system

$$\begin{cases} 2x + 5y - z = 1, \\ x - 2y + 3z = 2. \end{cases}$$

Sample problems

Problem 10. Factor a polynomial $p(x) = x^4 - 2x^3 - x^2 - 2x + 1$ into irreducible factors over the fields $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_5$ and $\mathbb{Z}_7$.

Problem 11. Let

$$M = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\}, \ \ J = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \,\middle|\, y \in \mathbb{R} \right\}.$$

(i) Show that M is a subring of the matrix ring $\mathcal{M}_{2,2}(\mathbb{R})$.
(ii) Show that J is a two-sided ideal in M.
(iii) Show that the factor ring M/J is isomorphic to $\mathbb{R} \times \mathbb{R}$.

Problem 12. The polynomial $f(x) = x^6 + 3x^5 - 5x^3 + 3x - 1$ has how many distinct complex roots?

Problem 6. Let G be the group of all symmetries of a regular tetrahedron T. The group G naturally acts on the set of vertices of T, the set of edges of T, and the set of faces of T.

(i) Show that each of the three actions is transitive.

(ii) Show that the stabilizer of any vertex is isomorphic to the symmetric group S_3.

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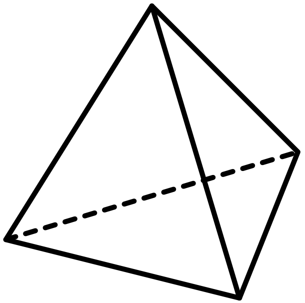

(i) Show that each of the three actions is transitive.

We can label vertices of T by 1, 2, 3 and 4. Then the action of G on the vertices induces a homomorphism $h : G \to S_4$ (permutation representation). This homomorphism is injective since any isometry of $\mathbb{R}^3$ is uniquely determined by images of any 4 points not in the same plane. Observe that every transposition is in the image $h(G)$ (it is realized by a reflection about a plane of symmetry of T). Since the symmetric group S_4 is generated by transpositions, it follows that $h(G) = S_4$. Hence h is an isomorphism.

In view of the isomorphism h, the action of G on vertices of T is essentially the natural action of S_4 on $\{1, 2, 3, 4\}$. Since any two vertices of T are endpoints of a unique edge and any three vertices are vertices of a unique face, the actions of G on edges and vertices of T are essentially the actions of S_4 on two-element and three-element subsets of $\{1, 2, 3, 4\}$. Transitivity of all three actions follows.

(ii) Show that the stabilizer of any vertex is isomorphic to the symmetric group S_3.

(iii) Show that the stabilizer of any edge is isomorphic to the Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(iv) Show that the stabilizer of any face is isomorphic to S_3.

Let $\alpha : G \curvearrowright X$ be an action of the group G on a set X. The stabilizer of a point $x \in X$ consists of all elements of G that fix x: $\mathrm{Stab}(x) = \{g \in G \mid gx = x\}$. It is a subgroup of G.

Suppose y is a point in the same orbit as x, that is, $y = hx$ for some $h \in G$. Then $g \in \mathrm{Stab}(y) \iff gy = y$ $\iff g(hx) = hx \iff h^{-1}(g(hx)) = h^{-1}(hx) \iff$ $(h^{-1}gh)x = (h^{-1}h)x = ex = x \iff h^{-1}gh \in \mathrm{Stab}(x)$.

It follows that $\mathrm{Stab}(y) = h\,\mathrm{Stab}(x)h^{-1}$. As a consequence, the group $\mathrm{Stab}(y)$ is isomorphic to $\mathrm{Stab}(x)$. In particular, if the action is transitive then the stabilizers of all points are isomorphic to one another.

(ii) Show that the stabilizer of any vertex is isomorphic to the symmetric group S_3.

(iii) Show that the stabilizer of any edge is isomorphic to the Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(iv) Show that the stabilizer of any face is isomorphic to S_3.

As shown before, the actions of the group G on vertices, edges and faces of the tetrahedron T are essentially the actions of the symmetric group S_4 on one-element, two-element and three element subsets of $\{1, 2, 3, 4\}$. "Essentially" means, in particular, that all stabilizers of the actions of G are isomorphic to stabilizers of the corresponding actions of S_4.

Since all three actions are transitive, it is enough to describe one stabilizer for each action. The stabilizer of 4 is naturally isomorphic to S_3, the group of all permutations on $\{1, 2, 3\}$. The stabilizer of the set $\{1, 2\}$ consists of id, $(1\,2)$, $(3\,4)$ and $(1\,2)(3\,4)$. It is isomorphic to the Klein 4-group. Finally, the stabilizer of the set $\{1, 2, 3\}$ coincides with the stabilizer of 4.

Problem 7. Let S be a nonempty set and $\mathcal{P}(S)$ be the set of all subsets of S. **(i)** Prove that $\mathcal{P}(S)$ with the operations of symmetric difference $\triangle$ (as addition) and intersection $\cap$ (as multiplication) is a commutative ring with unity.

(ii) Prove that the ring $\mathcal{P}(S)$ is isomorphic to the ring of functions $\mathcal{F}(S, \mathbb{Z}_2)$.

For any subset $E \subset S$ let $\chi_E : S \to \{0, 1\}$ be the characteristic function of E,

$$\chi_E(x) = \begin{cases} 1 & \text{if} \quad x \in E, \\ 0 & \text{if} \quad x \notin E. \end{cases}$$

Let E_1 and E_2 be any subsets of S. Note that $\chi_{E_1 \cap E_2}(x) = 1$ if and only if $\chi_{E_1}(x) = \chi_{E_2}(x) = 1$. Hence $\chi_{E_1 \cap E_2} = \chi_{E_1} \chi_{E_2}$. Since $E_1 \triangle E_2 = (E_1 \cup E_2) \setminus (E_1 \cap E_2)$, it follows that

$$\chi_{E_1 \triangle E_2} = \chi_{E_1} + \chi_{E_2} - 2\chi_{E_1 \cap E_2} \equiv \chi_{E_1} + \chi_{E_2} (\mathrm{mod}\, 2).$$

Let us consider the characteristic functions χ_E as taking their values in the ring $\mathbb{Z}_2$ rather than $\mathbb{R}$. This yields a map $F : \mathcal{P}(S) \to \mathcal{F}(S, \mathbb{Z}_2)$ given by $F(E) = \chi_E$ for all $E \subset S$.

By the above F is a homomorphism of the binary structure $\big(\mathcal{P}(S), \cap\big)$ to $\big(\mathcal{F}(S, \mathbb{Z}_2), \cdot\big)$ and, simultaneously, a homomorphism of $\big(\mathcal{P}(S), \triangle\big)$ to $\big(\mathcal{F}(S, \mathbb{Z}_2), +\big)$. The map F is clearly injective. It is also surjective since any function $g : S \to \mathbb{Z}_2$ can be represented as χ_E, where $E = \{x \in S \mid g(x) = 1\}$. Hence the homomorphism F is actually an isomorphism.

Since $\mathcal{F}(S, \mathbb{Z}_2)$ is a commutative ring with unity, the binary structure $\big(\mathcal{F}(S, \mathbb{Z}_2), +\big)$ is an abelian group while the binary structure $\big(\mathcal{F}(S, \mathbb{Z}_2), \cdot\big)$ is a commutative monoid. In view of the isomorphism F, the set $\mathcal{P}(S)$ is an abelian group relative to the operation $\triangle$ and a commutative monoid relative to the operation $\cap$. Moreover, $\cap$ distributes over $\triangle$, which follows (through F) from the distributive law in $\mathcal{F}(S, \mathbb{Z}_2)$. Thus $\big(\mathcal{P}(S), \triangle, \cap\big)$ is a commutative ring with unity and F is an isomorphism of rings.

Problem 8. Solve a system of congruences: $\begin{cases} x \equiv 2 \bmod 5, \\ x \equiv 3 \bmod 6, \\ x \equiv 6 \bmod 7. \end{cases}$

The moduli 5, 6 and 7 are pairwise coprime. By the generalized Chinese Remainder Theorem, all solutions of the system form a single congruence class modulo $5 \cdot 6 \cdot 7 = 210$. It remains to find a particular solution. One way to do this is to represent 1 as an integral linear combination of $6 \cdot 7 = 42$, $5 \cdot 7 = 35$ and $5 \cdot 6 = 30$ (note that 1 is the greatest common divisor of these numbers). Suppose $1 = 42n_1 + 35n_2 + 30n_3$ for some integers n_1, n_2, n_3. Then the numbers $x_1 = 42n_1$, $x_2 = 35n_2$ and $x_3 = 30n_3$ satisfy the following systems of congruences:

$$\begin{cases} x_1 \equiv 1 \bmod 5 \\ x_1 \equiv 0 \bmod 6 \\ x_1 \equiv 0 \bmod 7 \end{cases} \qquad \begin{cases} x_2 \equiv 0 \bmod 5 \\ x_2 \equiv 1 \bmod 6 \\ x_2 \equiv 0 \bmod 7 \end{cases} \qquad \begin{cases} x_3 \equiv 0 \bmod 5 \\ x_3 \equiv 0 \bmod 6 \\ x_3 \equiv 1 \bmod 7 \end{cases}$$

It follows that $x_0 = 2x_1 + 3x_2 + 6x_3$ is a solution of the given system.

Let us apply the generalized Euclidean algorithm (in matrix form) to 42, 35 and 30. We begin with the augmented matrix of a system

$$\begin{cases} y_1 = 42, \\ y_2 = 35, \\ y_3 = 30. \end{cases}$$

At each step, we choose two numbers in the rightmost column, divide the larger of them by the smaller, and replace the dividend with the remainder. This can be done by applying an elementary row operation to the matrix.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 42 \\ 0 & 1 & 0 & 35 \\ 0 & 0 & 1 & 30 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 12 \\ 0 & 1 & 0 & 35 \\ 0 & 0 & 1 & 30 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 12 \\ -2 & 1 & 2 & 11 \\ 0 & 0 & 1 & 30 \end{array}\right) \to \left(\begin{array}{ccc|c} 3 & -1 & -3 & 1 \\ -2 & 1 & 2 & 11 \\ 0 & 0 & 1 & 30 \end{array}\right)$$

The first row of the last matrix corresponds to a linear equation $3y_1 - y_2 - 3y_3 = 1$. By construction, $(y_1, y_2, y_3) = (42, 35, 30)$ is a solution of that equation. In other words,

$$1 = 42 \cdot 3 + 35 \cdot (-1) + 30 \cdot (-3).$$

Let x_1, x_2, x_3 be the terms in this expansion of 1: $x_1 = 42 \cdot 3 = 126$, $x_2 = 35 \cdot (-1) = -35$ and $x_3 = 30 \cdot (-3) = -90$. By the above, one solution of the given system of congruences is

$$\begin{aligned} x_0 &= 2x_1 + 3x_2 + 6x_3 \\ &= 2 \cdot 126 + 3(-35) + 6(-90) = -393. \end{aligned}$$

Another solution is $-393 + 2 \cdot 210 = 27$. The general solution is $x = 27 + 210n$, $n \in \mathbb{Z}$.

Problem 8. Solve a system of congruences: $\begin{cases} x \equiv 2 \bmod 5, \\ x \equiv 3 \bmod 6, \\ x \equiv 6 \bmod 7. \end{cases}$

Alternative solution. The general solution of the first congruence is $x = 2 + 5k$, $k \in \mathbb{Z}$. Substituting this into the second congruence, we obtain a linear congruence in k:

$$2 + 5k \equiv 3 \bmod 6 \iff 5k \equiv 1 \bmod 6$$

The multiplicative inverse of 5 modulo 6 is -1 since $5 \cdot (-1) = -5 \equiv 1 \bmod 6$. Hence the general solution of the linear congruence is $k = -1 + 6m$, where $m \in \mathbb{Z}$. Then $x = 2 + 5k = 2 + 5(-1 + 6m) = -3 + 30m$. Substituting this into the third congruence of the system, we obtain

$$-3 + 30m \equiv 6 \bmod 7 \iff 30m \equiv 9 \bmod 7$$
$$\iff 2m \equiv 2 \bmod 7 \iff m \equiv 1 \bmod 7$$

Hence $m = 1 + 7n$, where $n \in \mathbb{Z}$. Then $x = -3 + 30m$ $= -3 + 30(1 + 7n) = 27 + 210n$ is the general solution of the system.

Problem 9. Find all integer solutions of a system

$$\begin{cases} 2x + 5y - z = 1, \\ x - 2y + 3z = 2. \end{cases}$$

First we solve the second equation for x and substitute it into the first equation:

$$\begin{cases} 2(2y - 3z + 2) + 5y - z = 1, \\ x = 2y - 3z + 2 \end{cases} \iff \begin{cases} 9y - 7z = -3, \\ x = 2y - 3z + 2. \end{cases}$$

For any integer solution of the equation $9y - 7z = -3$, the number y is a solution of the linear congruence $9y \equiv -3 \bmod 7$. Solving the congruence, we obtain

$$9y \equiv -3 \bmod 7 \iff 2y \equiv 4 \bmod 7 \iff y \equiv 2 \bmod 7.$$

Hence $y = 2 + 7k$, where $k \in \mathbb{Z}$. Now we find z and x by back substitution: $z = (9y + 3)/7 = (9(2 + 7k) + 3)/7$ $= 3 + 9k$ and $x = 2y - 3z + 2 = 2(2 + 7k) - 3(3 + 9k) + 2$ $= -3 - 13k$. Note that z and x are integers for all $k \in \mathbb{Z}$.

Problem 10. Factor a polynomial $p(x) = x^4 - 2x^3 - x^2 - 2x + 1$ into irreducible factors over the fields $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_5$ and $\mathbb{Z}_7$.

First consider p as a function on $\mathbb{C}$. For any $x \neq 0$ we have

$$\frac{p(x)}{x^2} = x^2 - 2x - 1 - \frac{2}{x} + \frac{1}{x^2} = \left(x^2 + \frac{1}{x^2}\right) - 2\left(x + \frac{1}{x}\right) - 1.$$

Let $y = x + \frac{1}{x}$. Then $x^2 + \frac{1}{x^2} = y^2 - 2$. Consequently, $p(x)/x^2 = (y^2 - 2) - 2y - 1 = y^2 - 2y - 3 = (y - 3)(y + 1)$.
Then

$$\begin{aligned} p(x) &= x^2(y-3)(y+1) = x^2(x + x^{-1} - 3)(x + x^{-1} + 1) \\ &= (x^2 - 3x + 1)(x^2 + x + 1). \end{aligned}$$

We have obtained the above factorization of p as an equality of functions on $\mathbb{C} \setminus \{0\}$. Now we can check (by direct multiplication) that, in fact, it holds as an equality of polynomials over any field.

Problem 10. Factor a polynomial $p(x) = x^4 - 2x^3 - x^2 - 2x + 1$ into irreducible factors over the fields $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_5$ and $\mathbb{Z}_7$.

We already know that $p(x) = (x^2 - 3x + 1)(x^2 + x + 1)$ over any field. Depending on the field, any of the two quadratic factors either is irreducible (if it has no roots) or else splits as a product of two linear factors.

Over the field $\mathbb{C}$, the polynomial $x^2 - 3x + 1$ has roots $\alpha_{1,2} = \frac{1}{2}(3 \pm \sqrt{5})$ and the polynomial $x^2 + x + 1$ has roots $\beta_{1,2} = \frac{1}{2}(-1 \pm i\sqrt{3})$. Hence the factorization into irreducible factors over $\mathbb{C}$ is $p(x) = (x - \alpha_1)(x - \alpha_2)(x - \beta_1)(x - \beta_2)$. Note that the numbers β_1 and β_2 are not real while α_1 and α_2 are real but not rational. Since $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, it follows that over $\mathbb{R}$, the factorization is $p(x) = (x - \alpha_1)(x - \alpha_2)(x^2 + x + 1)$, and over $\mathbb{Q}$, it is $p(x) = (x^2 - 3x + 1)(x^2 + x + 1)$.

In the case of a finite field, we find roots by trying all elements of the field. We obtain that $p(x) = (x + 1)^2(x^2 + x + 1)$ over the field $\mathbb{Z}_5$ and $p(x) = (x^2 - 3x + 1)(x - 2)(x + 3)$ over $\mathbb{Z}_7$.

Problem 11. Let

$$M = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}, \quad J = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \;\middle|\; y \in \mathbb{R} \right\}.$$

(i) Show that M is a subring of the matrix ring $\mathcal{M}_{2,2}(\mathbb{R})$.

For any $x, y, z, x', y', z' \in \mathbb{R}$ we obtain

$$\begin{pmatrix} x & 0 \\ y & z \end{pmatrix} + \begin{pmatrix} x' & 0 \\ y' & z' \end{pmatrix} = \begin{pmatrix} x + x' & 0 \\ y + y' & z + z' \end{pmatrix},$$

$$\begin{pmatrix} x & 0 \\ y & z \end{pmatrix} - \begin{pmatrix} x' & 0 \\ y' & z' \end{pmatrix} = \begin{pmatrix} x - x' & 0 \\ y - y' & z - z' \end{pmatrix},$$

$$\begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \begin{pmatrix} x' & 0 \\ y' & z' \end{pmatrix} = \begin{pmatrix} xx' & 0 \\ yx' + zy' & zz' \end{pmatrix}.$$

It follows that the set M is closed under addition, subtraction and multiplication. Clearly, M is nonempty. Therefore M is a subring of the matrix ring $\mathcal{M}_{2,2}(\mathbb{R})$.

Problem 11. Let

$$M = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\}, \quad J = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \,\middle|\, y \in \mathbb{R} \right\}.$$

(ii) Show that J is a two-sided ideal in M.
(iii) Show that the factor ring M/J is isomorphic to $\mathbb{R} \times \mathbb{R}$.

Consider a map $\phi : M \to \mathbb{R} \times \mathbb{R}$ given for any $x, y, z \in \mathbb{R}$ by

$$\phi \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} = (x, z).$$

It follows from the solution of part **(i)** that the map ϕ is a homomorphism of rings. Observe that the kernel $\mathrm{Ker}(\phi)$ $= \phi^{-1}(0, 0)$ coincides with the set J. Therefore J is a two-sided ideal in M (since the kernel of any homomorphism is a two-sided ideal). By the Fundamental Theorem on Homomorphisms, the factor ring M/J is isomorphic to $\phi(M) = \mathbb{R} \times \mathbb{R}$.

Problem 12. The polynomial $f(x) = x^6 + 3x^5 - 5x^3 + 3x - 1$ has how many distinct complex roots?

Let $p \in \mathbb{C}[x]$ be a nonzero polynomial. We say that $\alpha \in \mathbb{C}$ is a root of p of multiplicity $k \geq 1$ if the polynomial is divisible by $(x-\alpha)^k$ but not divisible by $(x-\alpha)^{k+1}$. Equivalently, $p(x) = (x-\alpha)^k q(x)$ for some polynomial q such that $q(\alpha) \neq 0$. If this is the case then

$$\begin{aligned} p'(x) &= \big((x-\alpha)^k\big)' q(x) + (x-\alpha)^k q'(x) \\ &= k(x-\alpha)^{k-1} q(x) + (x-\alpha)^k q'(x) = (x-\alpha)^{k-1} r(x), \end{aligned}$$

where $r(x) = kq(x) + (x-\alpha)q'(x)$. Note that $r(x)$ is a polynomial and $r(\alpha) = kq(\alpha) \neq 0$. Hence α is a root of p' of multiplicity $k-1$ if $k > 1$ and not a root of p' if $k = 1$.

Problem 12. The polynomial $f(x) = x^6 + 3x^5 - 5x^3 + 3x - 1$ has how many distinct complex roots?

By the Fundamental Theorem of Algebra, any polynomial $p \in \mathbb{C}[x]$ of degree $n \geq 1$ can be represented as

$$p(x) = c(x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n),$$

where $c, \alpha_1, \dots, \alpha_n \in \mathbb{C}$ and $c \neq 0$. The numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ are roots of p, they need not be distinct. We have

$$p(x) = c(x - \beta_1)^{k_1}(x - \beta_2)^{k_2} \dots (x - \beta_m)^{k_m},$$

where $\beta_1, \dots, \beta_m$ are distinct roots of p and $k_1, \dots, k_m$ are their multiplicities. It follows from the above that

$$\gcd(p(x), p'(x)) = (x - \beta_1)^{k_1 - 1}(x - \beta_2)^{k_2 - 1} \dots (x - \beta_m)^{k_m - 1}.$$

As a consequence, the number of distinct roots of the polynomial p equals $\deg(p) - \deg(\gcd(p, p'))$.

Problem 12. The polynomial $f(x) = x^6 + 3x^5 - 5x^3 + 3x - 1$ has how many distinct complex roots?

Let's use the Euclidean algorithm to find the greatest common divisor of the polynomials $f(x) = x^6 + 3x^5 - 5x^3 + 3x - 1$ and $f'(x) = 6x^5 + 15x^4 - 15x^2 + 3$. First we divide f by f':

$$x^6+3x^5-5x^3+3x-1 = (6x^5+15x^4-15x^2+3)\left(\tfrac{1}{6}x+\tfrac{1}{12}\right) + r(x),$$

where $r(x) = -\frac{5}{4}x^4 - \frac{5}{2}x^3 + \frac{5}{4}x^2 + \frac{5}{2}x - \frac{5}{4}$. It is convenient to replace the remainder $r(x)$ by its scalar multiple $\tilde{r}(x) = -\frac{4}{5}r(x) = x^4 + 2x^3 - x^2 - 2x + 1$. Next we divide f' by $\tilde{r}$:

$$6x^5 + 15x^4 - 15x^2 + 3 = (x^4 + 2x^3 - x^2 - 2x + 1)(6x + 3).$$

Since f' is divisible by $\tilde{r}$, it follows that $\gcd(f, f') = \gcd(f', r)$ $= \gcd(f', \tilde{r}) = \tilde{r}$. Thus the number of distinct complex roots of the polynomial f equals $\deg(f) - \deg(\gcd(f, f')) = 6 - 4$ $= 2$.

Problem 12. The polynomial $f(x) = x^6 + 3x^5 - 5x^3 + 3x - 1$ has how many distinct complex roots?

As a follow-up to the solution, we can find the roots of the polynomial f. It follows from the solution that the polynomial $g = f/\gcd(f, f')$ has the same roots as f but, unlike f, all roots of g are simple (i.e., of multiplicity 1). Dividing f by $\tilde{r}(x) = x^4 + 2x^3 - x^2 - 2x + 1$, we obtain

$$x^6+3x^5-5x^3+3x-1 = (x^4+2x^3-x^2-2x+1)(x^2+x-1).$$

The polynomial $g(x) = x^2 + x - 1$ has two real roots $\beta_{1,2} = \frac{1}{2}(-1 \pm \sqrt{5})$. Therefore $f(x) = (x - \beta_1)^{k_1}(x - \beta_2)^{k_2}$, where k_1 and k_2 are positive integers, $k_1 + k_2 = 6$. Note that $\beta_1\beta_2 = -1$ (the constant term of g) and $\beta_1^{k_1}\beta_2^{k_2} = -1$ (the constant term of f). Then $\beta_1^{k_1-k_2} = (-1)^{k_2+1}$, a rational number. This suggests $k_1 - k_2 = 0$ (so that $k_1 = k_2 = 3$). We can check by direct multiplication that, indeed,

$$x^6+3x^5-5x^3+3x-1 = (x^2+x-1)^3 = (x-\beta_1)^3(x-\beta_2)^3.$$