MATH 415–502 Fall 2022

Homework assignment #1

Problem 1. Determine which of the following functions $f : X \to Y$ are properly defined, that is, $f(x)$ is determined uniquely for any $x \in X$ and belongs to Y. Briefly explain.

(i) $f : \mathbb{Q} \to \mathbb{Q}$, given by $f(p/q) = q/p$ for all $p, q \in \mathbb{Z}^+$.
(ii) $g : \mathbb{Q}^+ \to \mathbb{Q}^+$, given by $g(p/q) = q/p$ for all $p, q \in \mathbb{Z}^+$.
(iii) $h : \mathbb{Q}^+ \to \mathbb{Z}$, given by $h(p/q) = q - p$ for all $p, q \in \mathbb{Z}^+$.

Problem 2. For each of the following functions $f : X \to Y$ given by the same formula $f(x) = x^3 - 3x$ but with different choices of the domain $X \subset \mathbb{R}$ and codomain $Y \subset \mathbb{R}$, determine whether it is injective, surjective or bijective.

(i) $X = [-3, 3]$, $Y = \mathbb{R}$.
(ii) $X = Y = [-2, 2]$.
(iii) $X = [-1, 1]$, $Y = [-3, 3]$.

Problem 3. Let A_1, A_2, B_1 and B_2 be some sets (finite or infinite). Suppose that A_1 is of the same cardinality as A_2 and B_1 is of the same cardinality as B_2. Prove that the set $A_1 \times B_1$ is of the same cardinality as the set $A_2 \times B_2$.

Problem 4. For any set X let $\mathcal{P}(X)$ denote the set of all subsets of X (called the *power set* of X). Prove that $\mathcal{P}(X)$ is never of the same cardinality as X.

Problem 5. For any $x, y \in \mathbb{Z}^+$ let $x * y = z$, where z is the largest integer less than the product of x and y. Is $(\mathbb{Z}^+, *)$ a (properly defined) binary structure?

Problem 6. A binary operation $*$ on the set $\mathbb{Z}^+$ is defined by $x * y = 2^{xy}$ for all $x, y \in \mathbb{Z}^+$. Is this operation commutative? Is it associative?

Problem 7. Is there a binary operation on a set of two elements that is commutative but not associative?

Problem 8. How many different binary operations can be defined on a set of n elements? How many of those operations are commutative?

Problem 9. Let $(S, *)$ be a binary structure. An element $x \in S$ is called *idempotent* if $x * x = x$. Prove that the set of all idempotent elements is closed under the operation $*$ provided that $*$ is commutative and associative.

Problem 10. The following is a partially completed Cayley table for a certain associative operation. Complete the table. Briefly explain.

$*$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	d	c	d
d				