MATH 415–502 Fall 2022

Homework assignment #10

Problem 1 (2 pts). Let R_1 and R_2 be rings with unity.

(i) Suppose I_1 is a (two-sided) ideal in R_1 and I_2 is an ideal in R_2. Show that $I_1 \times I_2$ is an ideal in the ring $R_1 \times R_2$.

(ii) Suppose I is an ideal in $R_1 \times R_2$. Show that $I = I_1 \times I_2$, where I_1 is an ideal in R_1 and I_2 is an ideal in R_2.

Problem 2 (3 pts). It is known that all ideals of the ring $\mathbb{Z}_n$ are of the form $d\mathbb{Z}_n = \mathbb{Z}_n \cap d\mathbb{Z}$, where d is a divisor of n. For each divisor d of the number 24, answer the following questions.

(i) Does the ring $d\mathbb{Z}_{24}$ have divisors of zero?
(ii) Is $d\mathbb{Z}_{24}$ a field?
(iii) Does the factor ring $\mathbb{Z}_{24}/d\mathbb{Z}_{24}$ have divisors of zero?
(iv) Is $\mathbb{Z}_{24}/d\mathbb{Z}_{24}$ a field?

Problem 3. Let R be a commutative ring and I be an ideal in R. The *radical* of I in R, denoted $\sqrt{I}$, is the set of all elements $a \in R$ such that $a^n \in I$ for some integer $n \geq 1$ (where n may depend on a). Prove that $\sqrt{I}$ is also an ideal in R.

Problem 4. For each divisor d of the number 24, find the radical of the ideal $d\mathbb{Z}_{24}$ in the ring $\mathbb{Z}_{24}$.

Problem 5. The radical of the trivial ideal $\{0\}$ is called the *nilradical*. Find the nilradical of the ring $\mathbb{Z}_{600}$.

Problem 6 (2 pts). Let $\mathcal{M}_{2,2}(\mathbb{R})$ denote the ring of 2×2 matrices with real entries. Find a left ideal $I_L \subset \mathcal{M}_{2,2}(\mathbb{R})$ and a right ideal $I_R \subset \mathcal{M}_{2,2}(\mathbb{R})$ that are not two-sided ideals.