MATH 415–502 Fall 2022

Homework assignment #11

Problem 1 (2 pts). Consider the ring $\mathbb{Z} \times \mathbb{Z}$.

(i) Show that all ideals of $\mathbb{Z} \times \mathbb{Z}$ are of the form $m\mathbb{Z} \times n\mathbb{Z}$, where m and n are nonnegative integers. [Hint: use Problem 1 from HW #10.]

(ii) Determine which ideals of $\mathbb{Z} \times \mathbb{Z}$ are maximal.

(iii) Determine which ideals of $\mathbb{Z} \times \mathbb{Z}$ are prime.

Problem 2 (2 pts). Consider the ring $\mathcal{F}(S, \mathbb{F})$ of all functions $h : S \to \mathbb{F}$, where S is a nonempty set and $\mathbb{F}$ is a field.

(i) Given a nonempty subset $S_0 \subset S$, let I_{S_0} be the set of all functions $h : S \to \mathbb{F}$ such that $h(x) = 0$ for $x \in S_0$. Show that I_{S_0} is an ideal in $\mathcal{F}(S, \mathbb{F})$. [Hint: I_{S_0} is the kernel of a certain homomorphism.]

(ii) Show that for any point $x_0 \in S$, the ideal $I_{\{x_0\}}$ is maximal.

(iii) Prove that the factor ring $\mathcal{F}(S, \mathbb{F})/I_{\{x_0\}}$ is isomorphic to $\mathbb{F}$.

Problem 3. Suppose $\mathbb{F}_k$ is a finite field with k elements. Let $p(x) \in \mathbb{F}_k[x]$ be an irreducible polynomial of degree n. Find the number of elements in the field $\mathbb{F}_k[x]/p(x)\mathbb{F}_k[x]$.

Problem 4 (3 pts). Consider a ring of polynomials $\mathbb{Z}_2[x]$.

(i) Show that $p(x) = x^3 + x + 1$ and $q(x) = x^3 + x^2 + 1$ are the only irreducible polynomials of degree 3 in $\mathbb{Z}_2[x]$.

(ii) Show that $q(x) = p(x - 1)$ in $F[x]$ for any field F of characteristic 2.

(iii) Let $\beta = x + p(x)\mathbb{Z}_2[x]$. Show that $0, 1, \beta, \beta+1, \beta^2, \beta^2+1, \beta^2+\beta, \beta^2+\beta+1$ is a complete list of elements of the field $\mathbb{Z}_2[x]/p(x)\mathbb{Z}_2[x]$.

(iv) Show that $p(\beta) = p(\beta^2) = p(\beta^2 + \beta) = 0$ and $q(\beta + 1) = q(\beta^2 + 1) = q(\beta^2 + \beta + 1) = 0$.

Problem 5. Consider the ring $\mathbb{Z}[\sqrt{3}] = \{m + n\sqrt{3} \mid m, n \in \mathbb{Z}\}$. Prove that the function $\phi : \mathbb{Z}[\sqrt{3}] \to \mathbb{Z}[\sqrt{3}]$ given by $\phi(m + n\sqrt{3}) = m - n\sqrt{3}$ for all $m, n \in \mathbb{Z}$ is an automorphism of this ring.

Problem 6. A non-zero, non-unit element of an integral domain is called *irreducible* if it cannot be represented as a product of two non-units.

(i) Decompose a polynomial $p(x) = 4x^2 + 4x - 8$ as a product of irreducible factors in the ring $\mathbb{Q}[x]$.

(ii) Decompose the same polynomial as a product of irreducible factors in the ring $\mathbb{Z}[x]$.

[Hint: an irreducible element of a polynomial ring is not necessarily the same thing as an irreducible polynomial.]