MATH 415–502 Fall 2022

Homework assignment #3

Problem 1. List all subgroups of the group $(\mathbb{Z}_{10}, +_{10})$.

Problem 2. Let H be the subgroup of the additive group $\mathbb{R}$ generated by 1 and $\sqrt{2}$: $H = \langle 1, \sqrt{2} \rangle$. Prove that H is not cyclic.

Problem 3. Prove that the additive group $\mathbb{Q}$ cannot be generated by a finite set.

[Hint: common denominator.]

Problem 4. Suppose that a group G has only finitely many subgroups. Prove that G is finite.

[Hint: Any element generates a cyclic subgroup. Show that for any cyclic subgroup H (finite or infinite) there are only finitely many elements g such that $\langle g \rangle = H$.]

Problem 5. Let a and b be elements of a group G. Prove that the elements ab and ba have the same order.

Problem 6. Draw the Cayley (di)graph of the group $\mathbb{Z}_8$ with respect to a generating set $S = \{3, 4\}$.

Problem 7. Consider the following permutations in S_6:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}.$$

Compute permutations $\tau^2\sigma$, $\sigma^{-1}\tau\sigma$ and σ^{2022}. You can use two-row notation or disjoint cycle decomposition to express results.

Problem 8. Express the following permutations in S_8 as a product of disjoint cycles, and then as a product of transpositions:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}, \qquad \tau = (1\ 2)(4\ 7\ 8)(2\ 1)(7\ 2\ 6\ 1\ 5).$$

Problem 9. We know that two permutations $\sigma, \tau \in S_n$ commute if they are disjoint. Also, $\sigma\tau = \tau\sigma$ if σ and τ belong to the same cyclic subgroup of S_n. Find an example of permutations $\sigma, \tau \in S_n$ such that $\sigma\tau = \tau\sigma$ while σ and τ are neither disjoint nor in the same cyclic subgroup.

[Hint: there is an example with $n = 4$.]

Problem 10. Suppose that a permutation $\sigma \in S_n$, where $n \geq 3$, commutes with any other permutation on n symbols: $\sigma\tau = \tau\sigma$ for all $\tau \in S_n$. Prove that σ is the identity map.

[Hint: it is enough to consider the case when τ is a transposition.]