MATH 415–502 Fall 2022

Homework assignment #5

Problem 1. Prove that a subgroup H of a group G is normal (that is, $gH = Hg$ for all $g \in G$) if and only if $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.

Problem 2. Let $\phi : G \to H$ be a homomorphism of groups. Prove that ϕ is injective if and only if its kernel $\mathrm{Ker}(\phi)$ is trivial.

Problem 3. Show that the relation "is a normal subgroup of" is not transitive. In other words, find a group G with two subgroups H and K such that K is a normal subgroup of H, H is a normal subgroup of G, but K is not a normal subgroup of G.

Problem 4 (2 pts). Given an abelian group G, let $F(G)$ be the set of all elements of finite order in G.

(i) Prove that $F(G)$ is a subgroup of G.
(ii) Describe $F(G)$ in the case when $G = \mathbb{R}/\mathbb{Z}$.

Problem 5. Given a group G, an element $c \in G$ is called *central* if it commutes with any other element: $cg = gc$ for all $g \in G$. The set of all central elements is called the *center* of G and denoted $Z(G)$. Prove that $Z(G)$ is a normal subgroup of G.

Problem 6. Given two elements g and h of a group G, the element $[g, h] = ghg^{-1}h^{-1}$ is called their *commutator*. The subgroup of G generated by all commutators is called the *commutator* (or *derived*) *group* of G and denoted $[G, G]$ (or G'). Prove that $[G, G]$ is a normal subgroup of G.

Problem 7. Prove that the commutator group of the symmetric group S_n is the alternating group A_n. [Hint: show that the product of any two transpositions is a commutator.]

Problem 8 (2 pts). A group G is called *perfect* if $[G, G] = G$. Prove that the alternating group A_n is perfect for $n \geq 5$ (without using the fact that it is simple).