MATH 415–502 Fall 2022

Homework assignment #9

Problem 1 (2 pts). Let $\mathbb{F}$ be a field.

(i) Suppose x_1 and x_2 are two different elements of $\mathbb{F}$. Construct a polynomial $p \in \mathbb{F}[x]$ of degree 1 such that $p(x_1) = 1$ and $p(x_2) = 0$.

(ii) Suppose $x_1, x_2, \dots, x_n$ ($n \geq 2$) are distinct elements of $\mathbb{F}$. Construct a polynomial $p \in \mathbb{F}[x]$ of degree $n-1$ such that $p(x_1) = 1$ and $p(x_i) = 0$ for $2 \leq i \leq n$.

(iii) Suppose $x_1, x_2, \dots, x_n$ ($n \geq 2$) are distinct elements of $\mathbb{F}$. Prove that for any elements $y_1, y_2, \dots, y_n \in \mathbb{F}$ there exists a polynomial $f \in \mathbb{F}[x]$ of degree at most $n-1$ such that $f(x_i) = y_i$ for $1 \leq i \leq n$.

Problem 2 (2 pts). Let $\mathbb{F}$ be a field.

(i) Suppose $x_1, x_2, \dots, x_n$ are distinct zeros of a polynomial $f \in \mathbb{F}[x]$. Prove that $f(x)$ is divisible by $(x - x_1)(x - x_2)\dots(x - x_n)$ without using the Unique Factorization Theorem.

(ii) Prove that any nonzero polynomial $f \in \mathbb{F}[x]$ has at most $\deg(f)$ distinct zeros.

(iii) Prove that the polynomial in Problem 1(iii) is unique.

Problem 3. Find the partial quotient and the remainder when a polynomial $f(x) = x^{100}$ is divided by $g(x) = x^{11} - 1$ (over any field).

Problem 4. Find the remainder when a polynomial $f(x) = x^9 - 3x^8 + 2x^2 - 2$ is divided by $g(x) = x^2 - 4x + 3$ (over any field of characteristic different from 2).

Problem 5. Find all prime numbers p such that a polynomial $x^4 + x^3 + x^2 - x + 1$ is divisible by $x + 2$ in $\mathbb{Z}_p[x]$.

Problem 6. Factor a polynomial $f(x) = x^4 - 2x^2 - 1$ into irreducible factors over the fields $\mathbb{R}$ and $\mathbb{C}$.

Problem 7. Factor a polynomial $f(x) = 2x^3 + 3x^2 + 5x + 2$ into irreducible factors over the field $\mathbb{Q}$.

Problem 8. Factor a polynomial $f(x) = x^3 - 2x^2 - 4$ into irreducible factors over the fields $\mathbb{Z}_5$ and $\mathbb{Z}_7$.