MATH 415

Modern Algebra I

Lecture 7:
Cycle decomposition.
Order and sign of a permutation.

Permutations

Let X be a finite set. A **permutation** of X is a bijection from X to itself.

Two-row notation. $\pi = \begin{pmatrix} a & b & c & \dots \\ \pi(a) & \pi(b) & \pi(c) & \dots \end{pmatrix}$,

where $a, b, c, \dots$ is a list of all elements in the domain of π.

The set of all permutations of a finite set X is called the **symmetric group** on X. *Notation:* S_X, Σ_X, $\mathrm{Sym}(X)$.

The set of all permutations of $\{1, 2, \dots, n\}$ is called the **symmetric group on n symbols** and denoted S_n or $S(n)$.

Given two permutations π and σ, the composition $\pi\sigma$, defined by $\pi\sigma(x) = \pi(\sigma(x))$, is called the **product** of these permutations. In general, $\pi\sigma \neq \sigma\pi$, i.e., multiplication of permutations is not commutative. However, it is associative: $\pi(\sigma\tau) = (\pi\sigma)\tau$.

Cycles

A permutation π of a set X is called a **cycle** (or **cyclic**) of length r if there exist r distinct elements $x_1, x_2, \ldots, x_r \in X$ such that

$$\pi(x_1) = x_2,\ \pi(x_2) = x_3, \ldots,\ \pi(x_{r-1}) = x_r,\ \pi(x_r) = x_1,$$

and $\pi(x) = x$ for any other $x \in X$.

Notation. $\pi = (x_1\ x_2\ \ldots\ x_r)$.

The identity function is (the only) cycle of length 1. Any cycle of length 2 is called a **transposition**.

The inverse of a cycle is also a cycle of the same length. Indeed, if $\pi = (x_1\ x_2\ \ldots\ x_r)$, then $\pi^{-1} = (x_r\ x_{r-1}\ \ldots\ x_2\ x_1)$.

Cycle decomposition

Let π be a permutation of X. We say that π **moves** an element $x \in X$ if $\pi(x) \neq x$. Otherwise π **fixes** x.

Two permutations π and σ are called **disjoint** if the set of elements moved by π is disjoint from the set of elements moved by σ.

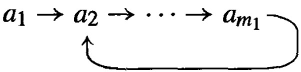
Theorem If π and σ are disjoint permutations in S_X, then they commute: $\pi\sigma = \sigma\pi$.

Idea of the proof: If π moves an element x, then it also moves $\pi(x)$. Hence σ fixes both so that $\pi\sigma(x) = \sigma\pi(x) = \pi(x)$.

Theorem Any permutation of a finite set can be expressed as a product of disjoint cycles. This **cycle decomposition** is unique up to rearrangement of the cycles involved.

Idea of the proof: Given $\pi \in S_X$, for any $x \in X$ consider a sequence $a_1 = x, a_2, a_3, \dots,$ where $a_{m+1} = \pi(a_m)$. Let r be the least index such that $a_r = a_k$ for some $k < r$. Then $k = 1$.

Cycle decomposition



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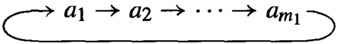

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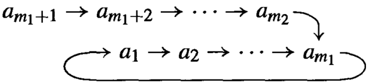

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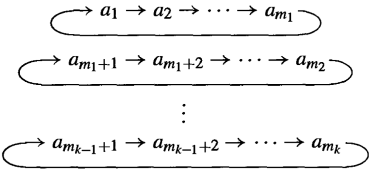

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Remark. Any cycle of length m can be denoted in m different ways depending on a choice of the initial point. For example, $(1\ 2\ 3\ 4) = (2\ 3\ 4\ 1) = (3\ 4\ 1\ 2) = (4\ 1\ 2\ 3)$.

Examples

- $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 4 & 7 & 9 & 1 & 12 & 5 & 11 & 3 & 10 & 6 & 8 \end{pmatrix}$

$= (1\ 2\ 4\ 9\ 3\ 7\ 5)(6\ 12\ 8\ 11)(10)$
$= (1\ 2\ 4\ 9\ 3\ 7\ 5)(6\ 12\ 8\ 11).$

- $(1\ 2)(2\ 3)(3\ 4)(4\ 5)(5\ 6) = (1\ 2\ 3\ 4\ 5\ 6).$

- $(1\ 2)(1\ 3)(1\ 4)(1\ 5) = (1\ 5\ 4\ 3\ 2).$

- $(2\ 4\ 3)(1\ 2)(2\ 3\ 4) = (1\ 4).$

Order of a permutation

The **order** of a permutation $\pi \in S_n$, denoted $|\pi|$ or $o(\pi)$, is defined as the smallest positive integer m such that $\pi^m = \mathrm{id}$, the identity map. In other words, this is the order of π as an element of the symmetric group S_n.

(Recall that every element of a finite group has finite order.)

Theorem Let π be a permutation of order m. Then $\pi^r = \pi^s$ if and only if $r \equiv s \bmod m$. In particular, $\pi^r = \mathrm{id}$ if and only if the order m divides r.

Remark. Notation $r \equiv s \bmod m$ (r is congruent to s modulo m) means that r and s leave the same remainder after division by m.

Theorem Let π be a cyclic permutation. Then the order $|\pi|$ equals the length of the cycle π.

Examples. • $\pi = (1\ 2\ 3\ 4\ 5)$.

$\pi^2 = (1\ 3\ 5\ 2\ 4),\ \pi^3 = (1\ 4\ 2\ 5\ 3),$
$\pi^4 = (1\ 5\ 4\ 3\ 2),\ \pi^5 = \mathrm{id}.$
$\implies |\pi| = 5.$

• $\sigma = (1\ 2\ 3\ 4\ 5\ 6)$.

$\sigma^2 = (1\ 3\ 5)(2\ 4\ 6),\ \sigma^3 = (1\ 4)(2\ 5)(3\ 6),$
$\sigma^4 = (1\ 5\ 3)(2\ 6\ 4),\ \sigma^5 = (1\ 6\ 5\ 4\ 3\ 2),\ \sigma^6 = \mathrm{id}.$
$\implies |\sigma| = 6.$

• $\tau = (1\ 2\ 3)(4\ 5)$.

$\tau^2 = (1\ 3\ 2),\ \tau^3 = (4\ 5),\ \tau^4 = (1\ 2\ 3),$
$\tau^5 = (1\ 3\ 2)(4\ 5),\ \tau^6 = \mathrm{id}.$
$\implies |\tau| = 6.$

Lemma 1 Let π and σ be two commuting permutations: $\pi\sigma = \sigma\pi$. Then
(i) the powers π^r and σ^s commute for all $r, s \in \mathbb{Z}$,
(ii) $(\pi\sigma)^r = \pi^r\sigma^r$ for all $r \in \mathbb{Z}$.

Lemma 2 Let π and σ be disjoint permutations in S_n. Then
(i) the powers π^r and σ^s are also disjoint,
(ii) $\pi^r\sigma^s = \mathrm{id}$ implies $\pi^r = \sigma^s = \mathrm{id}$.

Lemma 3 Let π and σ be disjoint permutations in S_n. Then
(i) they commute: $\pi\sigma = \sigma\pi$,
(ii) $(\pi\sigma)^r = \mathrm{id}$ if and only if $\pi^r = \sigma^r = \mathrm{id}$,
(iii) $|\pi\sigma| = \mathrm{lcm}\big(|\pi|, |\sigma|\big)$.

Theorem Let $\pi \in S_n$ and suppose that $\pi = \sigma_1\sigma_2\dots\sigma_k$ is a decomposition of π as a product of disjoint cycles. Then the order of π equals the least common multiple of the lengths of the cycles $\sigma_1, \dots, \sigma_k$.

Examples

- $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 4 & 7 & 9 & 1 & 12 & 5 & 11 & 3 & 10 & 6 & 8 \end{pmatrix}$.

The cycle decomposition is $\pi = (1\ 2\ 4\ 9\ 3\ 7\ 5)(6\ 12\ 8\ 11)$ or $\pi = (1\ 2\ 4\ 9\ 3\ 7\ 5)(6\ 12\ 8\ 11)(10)$. It follows that $|\pi| = \mathrm{lcm}(7, 4) = \mathrm{lcm}(7, 4, 1) = 28$.

- $\sigma = (1\ 2)(3\ 4)(5\ 6)$.

This permutation is a product of three disjoint transpositions. Therefore the order of σ equals $\mathrm{lcm}(2, 2, 2) = 2$.

- $\tau = (1\ 2)(1\ 3)(1\ 4)(1\ 5)$.

The permutation is given as a product of transpositions. However the transpositions are not disjoint and so this representation does not help to find the order of τ. The cycle decomposition is $\tau = (5\ 4\ 3\ 2\ 1)$. Hence τ is a cycle of length 5 so that $|\tau| = 5$.

Sign of a permutation

Theorem 1 (i) Any permutation of $n \geq 2$ elements is a product of transpositions. **(ii)** If $\pi = \tau_1\tau_2\ldots\tau_k = \tau'_1\tau'_2\ldots\tau'_m$, where τ_i, τ'_j are transpositions, then the numbers k and m are of the same parity (that is, both even or both odd).

A permutation π is called **even** if it is a product of an even number of transpositions, and **odd** if it is a product of an odd number of transpositions.

The **sign** $\operatorname{sgn}(\pi)$ of the permutation π is defined to be $+1$ if π is even, and -1 if π is odd.

Theorem 2 (i) $\operatorname{sgn}(\pi\sigma) = \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)$ for any $\pi, \sigma \in S_n$.
(ii) $\operatorname{sgn}(\pi^{-1}) = \operatorname{sgn}(\pi)$ for any $\pi \in S_n$.
(iii) $\operatorname{sgn}(\mathrm{id}) = 1$.
(iv) $\operatorname{sgn}(\tau) = -1$ for any transposition τ.
(v) $\operatorname{sgn}(\sigma) = (-1)^{r-1}$ for any cycle σ of length r.

Examples

- $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 4 & 7 & 9 & 1 & 12 & 5 & 11 & 3 & 10 & 6 & 8 \end{pmatrix}$.

First we decompose π into a product of disjoint cycles:

$$\pi = (1\ 2\ 4\ 9\ 3\ 7\ 5)(6\ 12\ 8\ 11).$$

The cycle $\sigma_1 = (1\ 2\ 4\ 9\ 3\ 7\ 5)$ has length 7, hence it is an even permutation. The cycle $\sigma_2 = (6\ 12\ 8\ 11)$ has length 4, hence it is an odd permutation. Then

$$\operatorname{sgn}(\pi) = \operatorname{sgn}(\sigma_1\sigma_2) = \operatorname{sgn}(\sigma_1)\operatorname{sgn}(\sigma_2) = 1 \cdot (-1) = -1.$$

- $\pi = (2\ 4\ 3)(1\ 2)(2\ 3\ 4)$.

π is represented as a product of cycles. The transposition has sign -1 while the cycles of length 3 have sign $+1$. Even though the cycles are not disjoint, $\operatorname{sgn}(\pi) = 1 \cdot (-1) \cdot 1 = -1$.