MATH 415

Modern Algebra I

Lecture 13: Transformation groups (continued). Group actions.

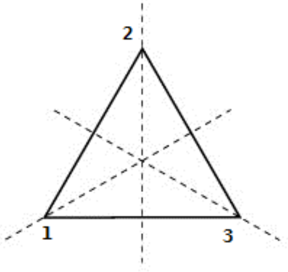
Groups of symmetries

Definition. A transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a **motion** (or a **rigid motion**) if it preserves distances between points.

Theorem All motions of $\mathbb{R}^n$ form a transformation group. Any motion $f : \mathbb{R}^n \to \mathbb{R}^n$ can be represented as $f(\mathbf{x}) = A\mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0 \in \mathbb{R}^n$ and A is an orthogonal matrix ($A^T A = AA^T = I$).

Given a geometric figure $F \subset \mathbb{R}^n$, a **symmetry** of F is a motion of $\mathbb{R}^n$ that preserves F. All symmetries of F form a transformation group.

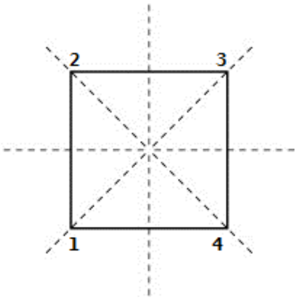
Example. • The **dihedral group** D_n is the group of symmetries of a regular n-gon. It consists of $2n$ elements: n reflections, $n-1$ rotations by angles $2\pi k/n$, $k = 1, 2, \ldots, n-1$, and the identity function.



Equlateral triangle

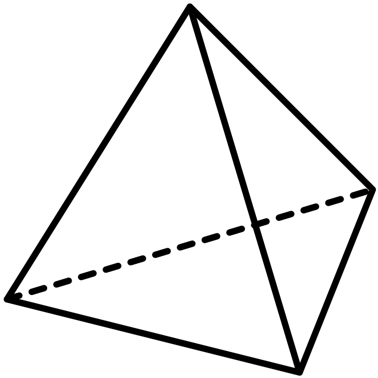
Any symmetry of a polygon maps vertices to vertices. Therefore it induces a permutation on the set of vertices. Moreover, the symmetry is uniquely recovered from the permutation.

In the case of the equilateral triangle, any permutation of vertices comes from a symmetry.



Square

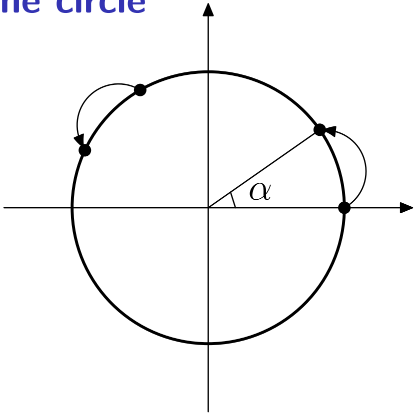
In the case of the square, not every permutation of vertices comes from a symmetry of the square. The reason is that a symmetry must map adjacent vertices to adjacent vertices.



Regular tetrahedron

Any symmetry of a polyhedron maps vertices to vertices. In the case of the regular tetrahedron, any permutation of vertices comes from a symmetry.

Rotations of the circle



Let $R_\alpha : S^1 \to S^1$ be the rotation of the circle S^1 by angle $\alpha \in \mathbb{R}$. All rotations R_α, $\alpha \in \mathbb{R}$ form a transformation group. Namely, $R_\alpha R_\beta = R_{\alpha+\beta}$, $R_\alpha^{-1} = R_{-\alpha}$, and $R_0 = \mathrm{id}$.

The group of rotations is a subgroup of the group of all symmetries of the circle (the other symmetries are reflections).

Group of automorphisms

Definition. Any isomorphism of a group G onto itself is called an **automorphism** of G.

Automorphisms are "symmetries" of the group as an algebraic structure. All automorphisms of a given group G form a transformation group denoted $\mathrm{Aut}(G)$.

Example. • Conjugation.

Take any $g \in G$ and define a map $i_g : G \to G$ by $i_g(x) = gxg^{-1}$ for all $x \in G$. Then $i_g(xy) = g(xy)g^{-1}$ $= gx(g^{-1}g)yg^{-1} = (gxg^{-1})(gyg^{-1}) = i_g(x)i_g(y)$. Hence i_g is a homomorphism. Further, $i_g(i_h(x)) = i_g(hxh^{-1})$ $= g(hxh^{-1})g^{-1} = (gh)x(gh)^{-1} = i_{gh}(x)$. Hence $i_g \circ i_h = i_{gh}$ for all $g, h \in G$. In particular, $i_g \circ i_{g^{-1}} = i_{g^{-1}} \circ i_g = i_e = \mathrm{id}_G$. Therefore $i_{g^{-1}} = (i_g)^{-1}$ so that i_g is bijective.

Automorphisms of the form i_g are called **inner**. They form a group $\mathrm{Inn}(G)$, which is a normal subgroup of $\mathrm{Aut}(G)$.

Group action

Definition. An **action** ϕ of a group G on a set X (denoted $\phi : G \curvearrowright X$) is a function $\phi : G \times X \to X$ such that

- $\phi(gh, x) = \phi(g, \phi(h, x))$ for all $g, h \in G$ and $x \in X$;
- $\phi(e, x) = x$ for all $x \in X$.

Typically, the element $\phi(g, x)$ is denoted gx. Then the above conditions can be rewritten as $g(hx) = (gh)x$ and $ex = x$.

The action ϕ can (and should) be regarded as a collection of transformations $T_g : X \to X$, $g \in G$, given by $T_g(x) = \phi(g, x)$. It follows from the definition that $T_g T_h = T_{gh}$, $T_{g^{-1}} = T_g^{-1}$, and $T_e = \mathrm{id}_X$. Hence $\{T_g\}_{g \in G}$ is a transformation group and $g \mapsto T_g$ is a homomorphism of the group G to the symmetric group S_X (called a **permutation representation**).

The group actions can be used to represent a given group as a transformation group or to parametrize a transformation group by an abstract group.

Examples of group actions

- Trivial action

Any group G acts on any nonempty set X; the action $\phi : G \curvearrowright X$ is given by $\phi(g, x) = x$.

- Scalar multiplication

The multiplicative group $\mathbb{R} \setminus \{0\}$ acts on any vector space V; the action $\phi : \mathbb{R} \setminus \{0\} \curvearrowright V$ is given by $\phi(\lambda, \mathbf{v}) = \lambda\mathbf{v}$.

- Natural action of a transformation group

G is a subgroup of S_X (all permutations of the set X); the action $\phi : G \curvearrowright X$ is given by $\phi(f, x) = f(x)$.

- Koopman representation

G is a subgroup of S_X; it acts on the vector space $\mathcal{F}(X, \mathbb{R})$ of functions $f : X \to \mathbb{R}$ by change of the variable. The action $\phi : G \curvearrowright \mathcal{F}(X, \mathbb{R})$ is given by $\phi(g, f) = f \circ g^{-1}$. Note that $(f \circ g_1^{-1}) \circ g_2^{-1} = f \circ (g_2 g_1)^{-1}$.

Examples of group actions

- Left adjoint action

Any group G acts on itself; the action $\phi : G \curvearrowright G$ is given by $\phi(g, x) = gx$.

- Right adjoint action

Any group G acts on itself; the action $\phi : G \curvearrowright G$ is given by $\phi(g, x) = xg^{-1}$. Note that $(xg_1^{-1})g_2^{-1} = x(g_2 g_1)^{-1}$.

- Conjugation

Any group G acts on itself; the action $\phi : G \curvearrowright G$ is given by $\phi(g, x) = gxg^{-1}$. This action is by automorphisms.

- Action on cosets of a subgroup

Any group G acts on the factor space G/H by a subgroup H (where H need not be normal); the action $\phi : G \curvearrowright G/H$ is given by $\phi(g, xH) = (gx)H$.

An action of the additive group $\mathbb{R}$ is called a **flow**.

Example. Consider an autonomous system of n ordinary differential equations of the first order

$$\begin{cases} \dot{x}_1 = g_1(x_1, x_2, \dots, x_n), \\ \dot{x}_2 = g_2(x_1, x_2, \dots, x_n), \\ \dots\dots\dots \\ \dot{x}_n = g_n(x_1, x_2, \dots, x_n), \end{cases}$$

where $g_1, g_2, \dots, g_n$ are differentiable functions defined in a domain $D \subset \mathbb{R}^n$. In vector form, $\dot{\mathbf{v}} = G(\mathbf{v})$, where $G : D \to \mathbb{R}^n$ is a vector field. Assume that for any $\mathbf{x} \in D$ the initial value problem $\dot{\mathbf{v}} = G(\mathbf{v})$, $\mathbf{v}(0) = \mathbf{x}$ has a unique solution $\mathbf{v}_{\mathbf{x}}(t)$, $t \in \mathbb{R}$. For any $t \in \mathbb{R}$ and $\mathbf{x} \in D$ let $F_t(\mathbf{x}) = \mathbf{v}_{\mathbf{x}}(t)$. Then the maps $F_t : D \to D$, $t \in \mathbb{R}$ describe evolution of a dynamical system governed by the ODEs.

Since the system of ODEs is autonomous, it follows that $F_t F_s = F_{t+s}$ for all $t, s \in \mathbb{R}$ so that $\phi(t, \mathbf{x}) = F_t(\mathbf{x})$ is a flow on D.

Orbits

Suppose $\phi : G \curvearrowright X$ is a group action. Consider a relation $\sim$ on the set X such that $x \sim y$ if and only if $x = gy$ for some $g \in G$.

Proposition The relation $\sim$ is an equivalence relation.

The equivalence class of a point $x \in X$ consists of all points of the form gx, $g \in G$. It is called the **orbit** of x under the action ϕ and denoted Gx or $\mathrm{Orb}_\phi(x)$.

The term "orbit" is motivated by the flows that describe celestial motions.

The action $\phi : G \curvearrowright X$ is called **transitive** if the entire set X forms a single orbit. For example, the adjoint actions of the group G on itself (both left and right) are transitive.

The extreme opposite of a transitive action is the trivial action, for which every point of X is a separate orbit.

Stabilizers

Suppose $\phi : G \curvearrowright X$ is a group action.

Given an element $g \in G$, let $\mathrm{Fix}_\phi(g) = \{x \in X \mid gx = x\}$. Elements of $\mathrm{Fix}_\phi(g)$ are called **fixed points** of g (with respect to the action ϕ).

Given a point $x \in X$, let $\mathrm{Stab}_\phi(x) = \{g \in G \mid gx = x\}$. Then $\mathrm{Stab}_\phi(x)$ is a subgroup of G called the **stabilizer** (or **isotropy group**) of x.

Theorem For any point $x \in X$, the number of elements in the orbit of x is equal to the index of its stabilizer:

$$|\mathrm{Orb}_\phi(x)| = (G : \mathrm{Stab}_\phi(x)).$$

Idea of the proof: $g_1 x = g_2 x \iff g_1$ and g_2 are in the same coset of the subgroup $\mathrm{Stab}_\phi(x)$.

Cayley's Theorem

Suppose $\phi : G \curvearrowright X$ is a group action.

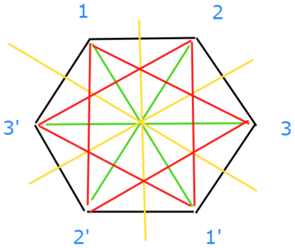
The action ϕ is called **faithful** if $T_g \neq T_h$ whenever $g \neq h$, where $T_g(x) = gx$. In other words, each element of G acts on X in a distinct way. In the case of a faithful action, the groups G and $\{T_g\}_{g\in G}$ are isomorphic. The action ϕ is called **free** if $\mathrm{Stab}_\phi(x) = \{e\}$ for all $x \in X$. It is called **totally non-free** if $\mathrm{Stab}_\phi(x) \neq \mathrm{Stab}_\phi(y)$ whenever $x \neq y$.

Theorem (Cayley) The left adjoint action of any group G is free and hence faithful. Consequently, any group is isomorphic to a transformation group.

Example. $\mathbb{R}$ with addition.

The left adjoint action is given by $\phi(g, x) = g + x$. The corresponding permutation representation is the group of translations of the real line $T_c(x) = x + c$, $c \in \mathbb{R}$.

Problem. Prove that $D_6 \cong S_3 \times \mathbb{Z}_2$.



The group D_6 is the group of symmetries of a regular hexagon. First we consider the action of D_6 on three long diagonals of the hexagon (green segments). After labeling those diagonals by 1, 2 and 3, it gives rise to a homomorphism $\phi : D_6 \to S_3$. Next we consider the action of D_6 on two equilateral triangles inscribed into the regular hexagon (red triangles). This gives rise to a homomorphism $\psi : D_6 \to \mathbb{Z}_2$. Finally, we define a map $f : D_6 \to S_3 \times \mathbb{Z}_2$ by $f(T) = (\phi(T), \psi(T))$. This map is also a homomorphism. We check that the kernel of f is trivial. Hence f is injective. As $|D_6| = |S_3 \times \mathbb{Z}_2| = 12$, we conclude that f is bijective.